Shape sensitivity analysis in discretized contact problems with a solution-dependent coefficient of friction

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History

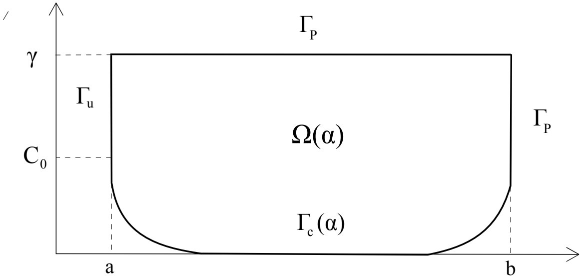
Shape optimization in discretized static contact problems with:

1. no friction [e.g. Haslinger, Neittaanmäki, 1996]
2. given friction (Tresca law) [e.g. Haslinger, Neittaanmäki, 1996]
3. Coulomb law - 2D case [Beremlijski, Haslinger, Kočvara, Outrata, 2002]
4. Coulomb law - 3D case [Beremlijski, Haslinger, Kočvara, Kučera, Outrata, 2009]

Goal

Extend the results to contact problems, where the coefficient of friction depends on the solution, i.e. the unknown displacement.

Geometrical setting



Elastic body and its contact boundary:

$$\Omega(\alpha) := \{(x_1, x_2) \mid a < x_1 < b,\ \alpha(x_1) < x_2 < \gamma\}, \quad \Gamma_c(\alpha) := \mathrm{Gr}\,\alpha,$$

where

$$\alpha \in U_{ad} := \{\alpha \in C^{1,1}([a,b]) \mid 0 \le \alpha \le C_0,\ |\alpha'| \le C_1,\ |\alpha''| \le C_2,\ \mathrm{meas}\,\Omega(\alpha) = C_3\}.$$

Classical formulation

Signorini problem

$$\operatorname{div}\sigma(\mathbf{u}) + \mathbf{F} = \mathbf{0} \quad \text{in } \Omega(\alpha),$$
$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_u,$$
$$(\,\sigma(\mathbf{u})n \equiv\,)\ \mathbf{T}(\mathbf{u}) = \mathbf{P} \quad \text{on } \Gamma_P,$$

$$\left.\begin{array}{rcl} (\,u_2(\cdot,\alpha(\cdot)) \equiv\,)\ u_2 \circ \alpha & \geq & -\alpha \\ T_2(\mathbf{u}) \circ \alpha & \geq & 0 \\ (u_2 \circ \alpha + \alpha)\, T_2(\mathbf{u}) \circ \alpha & = & 0 \end{array}\right\} \text{ in } (a,b)$$

Classical formulation

Signorini problem with given friction

$$\begin{aligned} \operatorname{div}\sigma(\mathbf{u}) + \mathbf{F} = \mathbf{0} &\quad \text{in } \Omega(\alpha), \\ \mathbf{u} = \mathbf{0} &\quad \text{on } \Gamma_u, \\ (\,\sigma(\mathbf{u})n \equiv\,)\ \mathbf{T}(\mathbf{u}) = \mathbf{P} &\quad \text{on } \Gamma_P, \end{aligned}$$

$$\left.\begin{array}{rcl} (\,u_2(\cdot,\alpha(\cdot)) \equiv\,)\ u_2 \circ \alpha & \geq & -\alpha \\ T_2(\mathbf{u}) \circ \alpha & \geq & 0 \\ (u_2 \circ \alpha + \alpha)\, T_2(\mathbf{u}) \circ \alpha & = & 0 \end{array}\right\} \text{ in } (a,b)$$

$$\left.\begin{array}{lll} u_1 = 0 & \Rightarrow & |T_1(\mathbf{u})| \leq \mathcal{F} \quad g \\ u_1 \neq 0 & \Rightarrow & T_1(\mathbf{u}) = -\operatorname{sgn}(u_1)\mathcal{F} \quad g \end{array}\right\} \text{ on } \Gamma_c(\alpha)$$

Classical formulation

Signorini problem with given friction and solution-dependent coefficient of friction:

$$\operatorname{div}\sigma(\mathbf{u}) + \mathbf{F} = \mathbf{0} \quad \text{in } \Omega(\alpha),$$
$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_u,$$
$$(\,\sigma(\mathbf{u})n \equiv\,)\ \mathbf{T}(\mathbf{u}) = \mathbf{P} \quad \text{on } \Gamma_P,$$

$$\left.\begin{array}{rcl} (\,u_2(\cdot,\alpha(\cdot)) \equiv\,)\ u_2 \circ \alpha & \geq & -\alpha \\ T_2(\mathbf{u}) \circ \alpha & \geq & 0 \\ (u_2 \circ \alpha + \alpha)\, T_2(\mathbf{u}) \circ \alpha & = & 0 \end{array}\right\} \text{ in } (a,b)$$

$$\left.\begin{array}{lll} u_1 = 0 & \Rightarrow & |T_1(\mathbf{u})| \leq \mathcal{F}(0)g \\ u_1 \neq 0 & \Rightarrow & T_1(\mathbf{u}) = -\operatorname{sgn}(u_1)\mathcal{F}(|u_1|)g \end{array}\right\} \text{ on } \Gamma_c(\alpha)$$

Classical formulation

Signorini problem with given friction and solution-dependent coefficient of friction:

$$\begin{aligned} \operatorname{div} \sigma(\mathbf{u}) + \mathbf{F} = \mathbf{0} \quad & \text{in } \Omega(\alpha), \\ \mathbf{u} = \mathbf{0} \quad & \text{on } \Gamma_u, \\ (\,\sigma(\mathbf{u})n \equiv\,)\ \mathbf{T}(\mathbf{u}) = \mathbf{P} \quad & \text{on } \Gamma_P, \end{aligned}$$

$$\left.\begin{array}{rcl} (\,u_2(\cdot,\alpha(\cdot)) \equiv\,)\ u_2 \circ \alpha & \geq & -\alpha \\ T_2(\mathbf{u}) \circ \alpha & \geq & 0 \\ (u_2 \circ \alpha + \alpha)\, T_2(\mathbf{u}) \circ \alpha & = & 0 \end{array}\right\} \text{ in } (a,b)$$

$$\left.\begin{array}{lll} u_1 = 0 & \Rightarrow & |T_1(\mathbf{u})| \leq \mathcal{F}(0) g \\ u_1 \neq 0 & \Rightarrow & T_1(\mathbf{u}) = -\operatorname{sgn}(u_1) \mathcal{F}(|u_1|) g \end{array}\right\} \text{ on } \Gamma_c(\alpha)$$

$$\begin{aligned} \sigma_{ij}(\mathbf{u}) = c_{ijkl}\varepsilon_{kl}(\mathbf{u}) \quad & \forall i,j = 1,2, \\ c_{ijkl} = c_{jikl} = c_{klij} \quad & \forall i,j,k,l = 1,2, \\ \exists C_{ell} > 0: \quad c_{ijkl}\xi_{ij}\xi_{kl} \geq C_{ell}\xi_{ij}\xi_{ij} \quad & \forall \xi_{ij} = \xi_{ji} \in \mathbb{R}. \end{aligned}$$

Variational formulation

Notation:

$$\mathbf{V}(\alpha) := \{ \mathbf{v} \in \mathbf{H}^1(\Omega(\alpha)) \mid \mathbf{v} = \mathbf{0} \ \text{ a.e. on } \Gamma_u \},$$
$$\mathbf{K}(\alpha) := \{ \mathbf{v} \in \mathbf{V}(\alpha) \mid v_2 \circ \alpha \geq -\alpha \ \text{ a.e. in } (a,b) \},$$
$$a(\mathbf{u},\mathbf{v}) := \int_{\Omega(\alpha)} c_{ijkl}\varepsilon_{kl}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\,dx; \quad L(\mathbf{v}) := \int_{\Omega(\alpha)} F_i v_i\,dx + \int_{\Gamma_P} P_i v_i\,ds.$$

Weak formulation of the state problem:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{K}(\alpha) \text{ such that:} \\ a(\mathbf{u},\mathbf{v}-\mathbf{u}) + \int_{\Gamma_c(\alpha)} \mathcal{F}(|u_1|)g(|v_1| - |u_1|)\,ds \geq L(\mathbf{v}-\mathbf{u}) \quad \forall \mathbf{v} \in \mathbf{K}(\alpha). \end{cases}$$

Existence, uniqueness, discretization ... [Haslinger, Vlach, 2005]

Derivation of the algebraic state problem

Let:

- $\mathcal{U}_{ad} \subset \mathbb{R}^p_+$ convex, compact,
- $\mathcal{K}(\boldsymbol{\alpha}) := \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}_\nu \geq -\boldsymbol{\alpha}\}$ for any $\alpha \in \mathcal{U}_{ad}$,

where the subvector $\mathbf{v}_\nu \in \mathbb{R}^p$ corresponds to normal displacement at the contact nodes.

Primal formulation of the state problem:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathcal{K}(\alpha) \text{ such that:} \\ \langle \mathbb{A}(\boldsymbol{\alpha})\mathbf{u}, \mathbf{v}-\mathbf{u}\rangle_n + \displaystyle\sum_{i=1}^{p} \omega_i(\boldsymbol{\alpha})\mathcal{F}(|(\mathbf{u}_\tau)_i|)\big(|(\mathbf{v}_\tau)_i| - |(\mathbf{u}_\tau)_i|\big) \\ \qquad \geq \langle \mathbf{L}(\boldsymbol{\alpha}), \mathbf{v}-\mathbf{u}\rangle_n \qquad\qquad \forall \mathbf{v} \in \mathcal{K}(\boldsymbol{\alpha}). \end{cases}$$

Derivation of the algebraic state problem

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Mixed formulation of the state problem:

$$\begin{cases} \text{Find } (\mathbf{u}, \boldsymbol{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^p_+ \text{ such that:} \\ \langle \mathbb{A}(\boldsymbol{\alpha})\mathbf{u}, \mathbf{v} - \mathbf{u}\rangle_n + \displaystyle\sum_{i=1}^{p} \omega_i(\boldsymbol{\alpha})\mathcal{F}(|(\mathbf{u}_\tau)_i|)\big(|(\mathbf{v}_\tau)_i| - |(\mathbf{u}_\tau)_i|\big) \\ \qquad \geq \langle \mathbf{L}(\boldsymbol{\alpha}), \mathbf{v} - \mathbf{u}\rangle_n + \langle \boldsymbol{\lambda}, \mathbf{v}_\nu - \mathbf{u}_\nu\rangle_p \quad \forall \mathbf{v} \in \mathbb{R}^n, \\ \langle \boldsymbol{\mu} - \boldsymbol{\lambda}, \mathbf{u}_\nu + \boldsymbol{\alpha}\rangle_p \geq 0 \quad \forall \boldsymbol{\mu} \in \mathbb{R}^p_+. \end{cases}$$

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- $\mathcal{K}(\boldsymbol{\alpha}) := \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}_\nu \geq -\boldsymbol{\alpha}\}$ for any $\alpha \in \mathcal{U}_{ad}$,

where the subvector $\mathbf{v}_\nu \in \mathbb{R}^p$ corresponds to normal displacement at the contact nodes.

Reduced algebraic state problem:

$$(\tilde{\mathcal{P}}(\boldsymbol{\alpha})) \quad \begin{cases} \text{Find } (\mathbf{u}_\tau, \mathbf{u}_\nu, \boldsymbol{\lambda}) \in \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^p_+ \text{ such that:} \\ \mathbf{0} \in \mathbb{A}_{\tau\tau}(\boldsymbol{\alpha})\mathbf{u}_\tau + \mathbb{A}_{\tau\nu}(\boldsymbol{\alpha})\mathbf{u}_\nu - \mathbf{L}_\tau(\boldsymbol{\alpha}) + \mathbf{Q}_1(\boldsymbol{\alpha}, \mathbf{u}_\tau), \\ \mathbf{0} = \mathbb{A}_{\nu\tau}(\boldsymbol{\alpha})\mathbf{u}_\tau + \mathbb{A}_{\nu\nu}(\boldsymbol{\alpha})\mathbf{u}_\nu - \boldsymbol{\lambda} - \mathbf{L}_\nu(\boldsymbol{\alpha}), \\ \mathbf{0} \in \mathbf{u}_\nu + \boldsymbol{\alpha} + N_{\mathbb{R}^p_+}(\boldsymbol{\lambda}), \end{cases}$$

where: $\quad \big(\mathbf{Q}_1(\boldsymbol{\alpha}, \mathbf{u}_\tau)\big)_i := \omega_i(\boldsymbol{\alpha})\mathcal{F}(|(\mathbf{u}_\tau)_i|)\partial|(\mathbf{u}_\tau)_i| \quad \forall i = 1, \ldots, p.$

State GE and its solvability

Introducing the state variable $\mathbf{y} := \left(\mathbf{u}_\tau, \mathbf{u}_\nu, \boldsymbol{\lambda}\right)^T$, we may rewrite $(\tilde{\mathcal{P}}(\boldsymbol{\alpha}))$ as:

$$\mathbf{0} \in \mathbf{F}(\boldsymbol{\alpha}, \mathbf{y}) + \mathbf{Q}(\boldsymbol{\alpha}, \mathbf{y}), \tag{GE}$$

where:

$$\mathbf{F}(\boldsymbol{\alpha}, \mathbf{y}) := \begin{pmatrix} \mathbb{A}_{\tau\tau}(\boldsymbol{\alpha}) & \mathbb{A}_{\tau\nu}(\boldsymbol{\alpha}) & \mathbf{0} \\ \mathbb{A}_{\nu\tau}(\boldsymbol{\alpha}) & \mathbb{A}_{\nu\nu}(\boldsymbol{\alpha}) & -\mathbb{I} \\ \mathbf{0} & \mathbb{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_3 \end{pmatrix} - \begin{pmatrix} \mathbf{L}_\tau(\boldsymbol{\alpha}) \\ \mathbf{L}_\nu(\boldsymbol{\alpha}) \\ -\boldsymbol{\alpha} \end{pmatrix}, \quad \mathbf{Q}(\boldsymbol{\alpha}, \mathbf{y}) := \begin{pmatrix} \mathbf{Q}_1(\boldsymbol{\alpha}, \mathbf{y}_1) \\ \mathbf{0} \\ N_{\mathbb{R}^p_+}(\mathbf{y}_3) \end{pmatrix}.$$

Theorem

Let the coefficient of friction $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}_+$ be Lipschitz continuous with a sufficiently small modulus. Then the mapping:

$$S : \boldsymbol{\alpha} \mapsto \{\mathbf{y} \mid \mathbf{0} \in \mathbf{F}(\boldsymbol{\alpha}, \mathbf{y}) + \mathbf{Q}(\boldsymbol{\alpha}, \mathbf{y})\}$$

is single-valued and Lipschitz continuous in $\mathcal{U}_{ad}$.

Let $J : (\boldsymbol{\alpha}, \mathbf{y}) \mapsto \mathbb{R}$ be a *continuously differentiable* cost functional. Then the shape optimization problem reads as:

$$(\mathbb{P}) \begin{cases} \text{minimize} & J(\boldsymbol{\alpha}, \mathbf{y}) \\ \text{subj. to} & \mathbf{0} \in \mathbf{F}(\boldsymbol{\alpha}, \mathbf{y}) + \mathbf{Q}(\boldsymbol{\alpha}, \mathbf{y}) \\ & \boldsymbol{\alpha} \in \mathcal{U}_{ad} \end{cases}$$

From now on let the assumptions of Theorem hold. Then:

Implicit Programming:

$$(\mathbb{P}) \quad \Longleftrightarrow \quad (\tilde{\mathbb{P}}) \begin{cases} \text{minimize} & \mathcal{J}(\boldsymbol{\alpha}) := J(\boldsymbol{\alpha}, S(\boldsymbol{\alpha})) \\ \text{subj. to} & \boldsymbol{\alpha} \in \mathcal{U}_{ad} \end{cases}$$

Given $\bar{\boldsymbol{\alpha}} \in \mathcal{U}_{ad}$, we need to compute:

- the solution to the GE: $\bar{\mathbf{y}} := S(\bar{\boldsymbol{\alpha}})$,
- one Clarke's subgradient: $\boldsymbol{\xi} \in \bar{\partial}\mathcal{J}(\bar{\boldsymbol{\alpha}}) = \nabla_\alpha J(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}}) + \left(\bar{\partial} S(\bar{\boldsymbol{\alpha}})\right)^T \nabla_y J(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}})$

For a set $A \subset \mathbb{R}^n$ and $\bar{\mathbf{x}} \in A$ denote by

$$\widehat{N}_A(\bar{\mathbf{x}}) := \left\{ \mathbf{x}^* \in \mathbb{R}^n \,\middle|\, \limsup_{\mathbf{x} \xrightarrow{A} \bar{\mathbf{x}}} \frac{\langle \mathbf{x}^*, \mathbf{x} - \bar{\mathbf{x}} \rangle_n}{\|\mathbf{x} - \bar{\mathbf{x}}\|_n} \leq 0 \right\} \quad \text{and} \quad N_A(\bar{\mathbf{x}}) := \operatorname*{Lim\,sup}_{\mathbf{x} \xrightarrow{A} \bar{\mathbf{x}}} \widehat{N}_A(\mathbf{x})$$

the regular (Fréchet) and limiting (Mordukhovich) normal cones to A at $\bar{\mathbf{x}}$, respectively.

For a multifunction $Q : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \operatorname{Gr} Q$ the multifunction $D^* Q(\bar{\mathbf{x}}, \bar{\mathbf{y}}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined by

$$D^* Q(\bar{\mathbf{x}}, \bar{\mathbf{y}})(\mathbf{y}^*) := \{\mathbf{x}^* \in \mathbb{R}^n \mid (\mathbf{x}^*, -\mathbf{y}^*) \in N_{\operatorname{Gr} Q}(\bar{\mathbf{x}}, \bar{\mathbf{y}})\}$$

is called the limiting (Mordukhovich) coderivative of Q at $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$.

Q is said to be calm at $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ provided $\exists L > 0$ $\exists$ neighbourhoods U, V of $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, resp.:

$$Q(\mathbf{x}) \cap V \subset Q(\bar{\mathbf{x}}) + L\|\mathbf{x} - \bar{\mathbf{x}}\|_n \mathbb{B}_m(\mathbf{0}, 1) \quad \forall \mathbf{x} \in U.$$

Sensitivity analysis - the adjoint GE

Lemma

$$D^*S(\bar{\boldsymbol{\alpha}})(\nabla_y J(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}})) \subset \left(\bar{\partial} S(\bar{\boldsymbol{\alpha}})\right)^T \nabla_y J(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}}) \quad \forall \bar{\boldsymbol{\alpha}} \in \mathcal{U}_{ad}.$$

... cf. [Mordukhovich, 1994]

$\Longrightarrow$ Our goal is to determine one element from $D^*S(\bar{\boldsymbol{\alpha}})(\nabla_y J(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}}))$!

Sensitivity analysis - the adjoint GE

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... cf. [Mordukhovich, 1994]

$\Longrightarrow$ Our goal is to determine one element from $D^*S(\bar{\alpha})(\nabla_y J(\bar{\alpha},\bar{\mathbf{y}}))$!

Theorem

(i) The multifunction:

$$M: \mathbf{p} \mapsto \{(\alpha,\mathbf{y}) \mid \mathbf{p} + \Phi(\alpha,\mathbf{y}) \in \operatorname{Gr} Q\},$$

where $\Phi(\alpha,\mathbf{y}) := \left(\alpha,\mathbf{y},-\mathbf{F}(\alpha,\mathbf{y})\right)^T$ is calm at $(\mathbf{0},\bar{\alpha},\bar{\mathbf{y}})$.

(ii) For each $\mathbf{p}^* \in D^*\mathcal{S}(\bar{\alpha})(\nabla_y J(\bar{\alpha},\bar{\mathbf{y}}))$ there exists a vector $\mathbf{v}^*$ such that:

$$\begin{pmatrix} \mathbf{p}^* \\ -\nabla_y J(\bar{\alpha},\bar{\mathbf{y}}) \end{pmatrix} \in \nabla \mathbf{F}(\bar{\alpha},\bar{\mathbf{y}})^T \mathbf{v}^* + D^*\mathbf{Q}(\bar{\alpha},\bar{\mathbf{y}},-\mathbf{F}(\bar{\alpha},\bar{\mathbf{y}}))(\mathbf{v}^*). \qquad \text{(AGE)}$$

... for (ii) cf. [Kočvara, Outrata, 2004]

Sensitivity analysis - adjoint GE

Sketch of the proof of (i):

By contradiction:

- Show that noncalmness of M at $(\mathbf{0}, \bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}})$ implies noncalmness of

$$\widetilde{M} : \tilde{\mathbf{p}} \mapsto \{(\boldsymbol{\alpha}, \mathbf{y}) \mid (\mathbf{0}, \mathbf{0}, \tilde{\mathbf{p}}) + \Phi(\boldsymbol{\alpha}, \mathbf{y}) \in \operatorname{Gr} Q\}$$

  at $(\mathbf{0}, \bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}})$;

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  at $(\mathbf{0}, \bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}})$;

- Observe, that: $(\boldsymbol{\alpha}, \mathbf{y}) \in \widetilde{M}(\tilde{\mathbf{p}}) \Leftrightarrow \tilde{\mathbf{p}} \in F(\boldsymbol{\alpha}, \mathbf{y}) + Q(\boldsymbol{\alpha}, \mathbf{y}) \Leftrightarrow \mathbf{0} \in \widetilde{F}(\boldsymbol{\alpha}, \tilde{\mathbf{y}}) + Q(\boldsymbol{\alpha}, \tilde{\mathbf{y}})$, where $\tilde{\mathbf{y}} := (\mathbf{y}_1,\ \mathbf{y}_2 - \tilde{\mathbf{p}}_2,\ \mathbf{y}_3)^T$ and

$$\widetilde{F}(\boldsymbol{\alpha}, \tilde{\mathbf{y}}) := \begin{pmatrix} \mathbb{A}_{\tau\tau}(\boldsymbol{\alpha}) & \mathbb{A}_{\tau\nu}(\boldsymbol{\alpha}) & \mathbf{0} \\ \mathbb{A}_{\nu\tau}(\boldsymbol{\alpha}) & \mathbb{A}_{\nu\nu}(\boldsymbol{\alpha}) & -\mathbb{I} \\ \mathbf{0} & \mathbb{I} & \mathbf{0} \end{pmatrix} \tilde{\mathbf{y}} - \begin{pmatrix} \mathbf{L}_\tau(\boldsymbol{\alpha}) + \tilde{\mathbf{p}}_1 - \mathbb{A}_{\tau\nu}(\boldsymbol{\alpha})\tilde{\mathbf{p}}_3 \\ \mathbf{L}_\nu(\boldsymbol{\alpha}) + \tilde{\mathbf{p}}_2 - \mathbb{A}_{\nu\nu}(\boldsymbol{\alpha})\tilde{\mathbf{p}}_3 \\ -\boldsymbol{\alpha} \end{pmatrix},$$

  i.e. $\tilde{\mathbf{y}}$ is the unique solution to the state problem on the domain $\Omega(\alpha)$, but with a different load vector;

Sensitivity analysis - adjoint GE

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  i.e. $\tilde{\mathbf{y}}$ is the unique solution to the state problem on the domain $\Omega(\alpha)$, but with a different load vector;

- Show that on a fixed domain the solution $\mathbf{y}$ depends Lipschitz continuously on the load vector $\mathbf{L}$;

Sketch of the proof of (i):

By contradiction:

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  at $(\mathbf{0}, \bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}})$;

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$$\widetilde{F}(\boldsymbol{\alpha}, \tilde{\mathbf{y}}) := \begin{pmatrix} \mathbb{A}_{\tau\tau}(\boldsymbol{\alpha}) & \mathbb{A}_{\tau\nu}(\boldsymbol{\alpha}) & \mathbf{0} \\ \mathbb{A}_{\nu\tau}(\boldsymbol{\alpha}) & \mathbb{A}_{\nu\nu}(\boldsymbol{\alpha}) & -\mathbb{I} \\ \mathbf{0} & \mathbb{I} & \mathbf{0} \end{pmatrix} \tilde{\mathbf{y}} - \begin{pmatrix} \mathbf{L}_\tau(\boldsymbol{\alpha}) + \tilde{\mathbf{p}}_1 - \mathbb{A}_{\tau\nu}(\boldsymbol{\alpha})\tilde{\mathbf{p}}_3 \\ \mathbf{L}_\nu(\boldsymbol{\alpha}) + \tilde{\mathbf{p}}_2 - \mathbb{A}_{\nu\nu}(\boldsymbol{\alpha})\tilde{\mathbf{p}}_3 \\ -\boldsymbol{\alpha} \end{pmatrix},$$

  i.e. $\tilde{\mathbf{y}}$ is the unique solution to the state problem on the domain $\Omega(\alpha)$, but with a different load vector;

- Show that on a fixed domain the solution $\mathbf{y}$ depends Lipschitz continuously on the load vector $\mathbf{L}$;
- From these facts prove that $\widetilde{M}$ is calm at $(\mathbf{0}, \bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}})$.

Sensitivity analysis - computation of D^*Q

Recall:

$$\mathbf{Q}(\boldsymbol{\alpha}, \mathbf{y}) = \begin{pmatrix} \mathbf{Q}_1(\boldsymbol{\alpha}, \mathbf{y}_1) \\ 0 \\ N_{\mathbb{R}^p_+}(\mathbf{y}_3) \end{pmatrix} \quad \ldots \quad \text{components are decoupled},$$

therefore its coderivative can be computed componentwise:

$$D^*\mathbf{Q}(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}}, \bar{\mathbf{q}})(\mathbf{q}^*) = \begin{pmatrix} D^*\mathbf{Q}_1(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}}_1, \bar{\mathbf{q}}_1)(\mathbf{q}_1^*) \\ 0 \\ D^* N_{\mathbb{R}^p_+}(\bar{\mathbf{y}}_3, \bar{\mathbf{q}}_3)(\mathbf{q}_3^*) \end{pmatrix} \quad \forall (\bar{\boldsymbol{\alpha}}, \bar{\mathbf{y}}, \bar{\mathbf{q}}) \in \operatorname{Gr}\mathbf{Q} \ \forall \mathbf{q}^*.$$

3^{rd} component is easy and well-known,

1^{st} component is much more challenging!

Write $\mathbf{Q}_1$ as a composition of an inner smooth mapping and an outer multifunction:

$$\mathbf{Q}_1(\boldsymbol{\alpha}, \mathbf{u}) = \begin{pmatrix} \omega_1(\boldsymbol{\alpha})\mathcal{F}(|u_1|)\partial|u_1| \\ \vdots \\ \omega_p(\boldsymbol{\alpha})\mathcal{F}(|u_p|)\partial|u_p| \end{pmatrix} = \begin{pmatrix} Z \circ \Psi_1 \\ \vdots \\ Z \circ \Psi_p \end{pmatrix}(\boldsymbol{\alpha}, \mathbf{u}) = (\mathbf{Z}_1 \circ \Psi)(\boldsymbol{\alpha}, \mathbf{u}),$$

where: $\Psi_i : (\boldsymbol{\alpha}, \mathbf{u}) \mapsto (\omega_i(\boldsymbol{\alpha}), u_i)^T$ and $Z : (x_1, x_2)^T \mapsto x_1\mathcal{F}(|x_2|)\partial|x_2|$.

Sensitivity analysis - computation of D^*Q_1

Write $\mathbf{Q}_1$ as a composition of an inner smooth mapping and an outer multifunction:

$$\mathbf{Q}_1(\boldsymbol{\alpha}, \mathbf{u}) = \begin{pmatrix} \omega_1(\boldsymbol{\alpha})\mathcal{F}(|u_1|)\partial|u_1| \\ \vdots \\ \omega_p(\boldsymbol{\alpha})\mathcal{F}(|u_p|)\partial|u_p| \end{pmatrix} = \begin{pmatrix} Z \circ \Psi_1 \\ \vdots \\ Z \circ \Psi_p \end{pmatrix} (\boldsymbol{\alpha}, \mathbf{u}) = (\mathbf{Z}_1 \circ \Psi)(\boldsymbol{\alpha}, \mathbf{u}),$$

where: $\Psi_i : (\boldsymbol{\alpha}, \mathbf{u}) \mapsto (\omega_i(\boldsymbol{\alpha}), u_i)^T$ and $Z : (x_1, x_2)^T \mapsto x_1 \mathcal{F}(|x_2|)\partial|x_2|$.

Hence:

$$D^*\mathbf{Q}_1(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{u}}, \bar{\mathbf{q}})(\mathbf{q}^*) \subset \nabla\Psi(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{u}})^T D^*\mathbf{Z}_1(\Psi(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{u}}), \bar{\mathbf{q}})(\mathbf{q}^*),$$

at all points $(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{u}}, \bar{\mathbf{q}}) \in \operatorname{Gr} \mathbf{Q}_1$ for which the following holds:

$$\operatorname{Ker} \nabla\Psi(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{u}})^T \cap D^*\mathbf{Z}_1(\Psi(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{u}}), \bar{\mathbf{q}})(0) = \{0\} \; ! \qquad \text{(CQ)}$$

Sensitivity analysis - final step

Notice, that the components of $\mathbf{Z}_1$ are also decoupled, therefore the coderivative $D^*\mathbf{Z}_1$ may be computed componentwise, i.e. in terms of D^*Z!

The quantity $D^*Z(\bar{x}_1, \bar{x}_2, \bar{z})(z^*)$ may be expressed in terms of the data of our problem at all reference points $(\bar{x}_1, \bar{x}_2, \bar{z}) \in \operatorname{Gr} Z$ and for all directions $z^* \in \mathbb{R}$.

Different parts of $\operatorname{Gr} Z$ correspond the different mechanical regimes:

- $|\bar{x}_2| > 0 \quad \Rightarrow \quad$ sliding,
- $\bar{x}_2 = 0,\ |z| < \bar{x}_1 \mathcal{F}(0) \quad \Rightarrow \quad$ strong sticking,
- $\bar{x}_2 = 0,\ |z| = \bar{x}_1 \mathcal{F}(0) \quad \Rightarrow \quad$ weak sticking.

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Corollary

(CQ) *is satisfied for all* $(\bar{\boldsymbol{\alpha}}, \bar{\mathbf{u}}, \bar{\mathbf{q}}) \in Gr\,\mathbf{Q}_1$.

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Thank you for your attention...