Optimal error estimates of DG time discretization for singularly perturbed problems

Miloslav Vlasák

Charles University Prague
Faculty of Mathematics and Physics

Motivation

- [Kaland, Roos, 2010]:
  suboptimal error estimates derived for singular perturbed problem
  $\|e\|_{L^\infty(L^2)} = O((N^{-1}\ln N)^2/\tau^{1/2} + \tau^{q+1/2} + N\tau^{q+1})$

Motivation

- [Kaland, Roos, 2010]:
  suboptimal error estimates derived for singular perturbed problem
  $\|e\|_{L^\infty(L^2)} = O((N^{-1}\ln N)^2/\tau^{1/2} + \tau^{q+1/2} + N\tau^{q+1})$
- Desired estimate:
  $\|e\|_{L^\infty(L^2)} = O((N^{-1}\ln N)^2 + \tau^{q+1})$

Singularly perturbed problem

$$\begin{aligned}
\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial x} + cu &= g, \quad \forall x \in (0,1), t \in (0,T), \quad (1) \\
u(0,t) = u(1,t) &= 0, \quad \forall t \in (0,T), \\
u(x,0) &= u^0(x), \quad \forall x \in (0,1),
\end{aligned}$$

Singularly perturbed problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu &= g, \quad \forall x \in (0,1), t \in (0,T), \quad (1)\\
u(0,t) = u(1,t) &= 0, \quad \forall t \in (0,T),\\
u(x,0) &= u^0(x), \quad \forall x \in (0,1),
\end{aligned}
$$

- $g \in L^2((0,1) \times (0,T))$
- $u^0 \in L^2(0,1)$
- $0 < \varepsilon << 1$
- functions b and c are sufficiently smooth with $b(x) > \beta > 0$ $c - \frac{1}{2}\frac{\partial b}{\partial x}(x) \geq c_0 > 0$

Weak formulation

Bilinear form:

$$a(u, v) = \varepsilon(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}) + (b\frac{\partial u}{\partial x} + cu, v)$$

Energy norm:

$$\|v\|_\varepsilon^2 = \|v\|^2 + \varepsilon|v|_{H^1(0,1)}^2$$

$$a(v, v) \geq \min(c_0, 1)\|v\|_\varepsilon^2 \geq 0$$

Weak formulation

Bilinear form:

$$a(u,v) = \varepsilon(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}) + (b\frac{\partial u}{\partial x} + cu, v)$$

Energy norm:

$$\|v\|_\varepsilon^2 = \|v\|^2 + \varepsilon|v|_{H^1(0,1)}^2$$
$$a(v,v) \geq \min(c_0, 1)\|v\|_\varepsilon^2 \geq 0$$

We say that the function $u \in C^1(0, T, H_0^1(0,1))$ is the weak solution, if

$$\begin{aligned} \left(\frac{\partial u(t)}{\partial t}, v\right) + a(u(t), v) &= (g(t), v), \quad \forall v \in H_0^1(0,1), \\ u(0) &= u^0. \end{aligned}$$

Properties of the exact solution

- Solution has in general boundary layer at $x = 1$.
- Assuming sufficiently compatible data solution has no interior layer.
- Moreover, it is possible to prove

$$\left|\frac{\partial^{k+m}u(x,t)}{\partial^k x \partial^m t}\right| \leq C\left(1 + \frac{1}{\varepsilon^k e^{\beta(1-x)/\varepsilon}}\right) \tag{2}$$

Space discretization

- Shishkin mesh: equidistantly distributed mesh points in intervals $[0, \sigma]$ and $[\sigma, 1]$, $\sigma = \frac{5}{2}\varepsilon \log(N)$.

Space discretization

- Shishkin mesh: equidistantly distributed mesh points in intervals $[0, \sigma]$ and $[\sigma, 1]$, $\sigma = \frac{5}{2}\varepsilon \log(N)$.
- N number of mesh points

Space discretization

- Shishkin mesh: equidistantly distributed mesh points in intervals $[0, \sigma]$ and $[\sigma, 1]$, $\sigma = \frac{5}{2}\varepsilon \log(N)$.
- N number of mesh points
- $V_N \subset H_0^1(0,1)$ contains piecewise linear functions

Space discretization

- Shishkin mesh: equidistantly distributed mesh points in intervals $[0, \sigma]$ and $[\sigma, 1]$, $\sigma = \frac{5}{2}\varepsilon \log(N)$.
- N number of mesh points
- $V_N \subset H_0^1(0,1)$ contains piecewise linear functions

Semidiscrete problem: find $u_N \in C^1(0, T, V_N)$ satisfying

$$\begin{aligned}(\frac{\partial u_N(t)}{\partial t}, v) + a(u_N(t), v) &= (g(t), v), \quad \forall v \in V_n, \forall t \in (0, T), \\ (u_N(0), v) &= (u^0, v). \quad \forall v \in V_N\end{aligned}$$

Time discretization

- Let $0 = t_0 < \ldots < t_r = T$ be a partition of $[0, T]$ and $I_m = (t_{m-1}, t_m)$.
- Let $\tau_m = |I_m|$ and $\tau = \max_m \tau_m$.
- $V_N^\tau = \{v \in L^2(0, T, V_N) : v|_{I_m} \in P^q(I_m, V_N)\}$
- $v \in V_N^\tau$: $v_\pm^m = v(t_m\pm) = \lim_{t\to t_m\pm} v(t)$, $\{v\}_m = v_+^m - v_-^m$

Definition

We say that the function $U \in V_N^\tau$ is the approximate solution to singularly perturbed problem (1) if

$$\begin{aligned} \int_{I_m} (U', v) + a(U, v)\mathrm{d}t + (\{U\}_{m-1}, v_+^{m-1}) &= \int_{I_m} (g, v)\mathrm{d}t, \\ &\qquad \forall v \in V_N^\tau,\ \forall m \\ (U_-^0, v) &= (u^0, v). \end{aligned}$$

Ritz projection

- Ritz projection: $R : H_0^1(0,1) \to V_N$, such that $a(u,v) = a(Ru,v)$, $\forall v \in V_N$

Ritz projection

- Ritz projection: $R : H_0^1(0,1) \to V_N$, such that $a(u,v) = a(Ru,v), \forall v \in V_N$
- It is possible to prove following estimates:
  $\|Ru - u\|_\varepsilon \leq C(N^{-1} \ln N)$
  $\|Ru - u\| \leq C(N^{-1} \ln N)^2$

Time interpolation

- Let us assume Radau quadrature nodes in interval I_m:
  $t_{m-1} < t_{m,q} < \ldots < t_{m,0} = t_m$
- $X^\tau = \{v \in L^2(0, T, L^2(0,1)) : v|_{I_m} \in P^q(I_m, L^2(0,1))\}$
- Time projection: $P_\tau : C([0, T], L^2(0,1)) \to X^\tau$, such that

$$P_\tau u(t_{m,i}) \quad = \quad u(t_{m,i})$$

Time interpolation

- Let us assume Radau quadrature nodes in interval I_m: $t_{m-1} < t_{m,q} < \ldots < t_{m,0} = t_m$
- $X^\tau = \{v \in L^2(0,T,L^2(0,1)) : v|_{I_m} \in P^q(I_m, L^2(0,1))\}$
- Time projection: $P_\tau : C([0,T], L^2(0,1)) \to X^\tau$, such that

$$P_\tau u(t_{m,i}) \quad = \quad u(t_{m,i})$$

Then

$$\sup_{I_m} \|P_\tau u - u\| \quad \leq \quad C\tau^{q+1}$$

Space-time projection

- Space-time projection: $\pi = RP_\tau = P_\tau R$

Space-time projection

- Space-time projection: $\pi = RP_\tau = P_\tau R$
- $\sup_{I_m} \|\pi u - u\| \leq C(\tau^{q+1} + (N^{-1} \ln N)^2)$

Radau quadrature

- Let $f \in C(0,1)$, we define Radau quadrature:

$$\int_{I_m} f(t)dt \approx Q[f] = \sum_{i=0}^{q} w_i f(t_{m,i})$$

- Radau quadrature has algebraic degree $2q$
- $0 < w_i \leq \tau_m$

Radau quadrature

- Let $f \in C(0,1)$, we define Radau quadrature:

$$\int_{I_m} f(t)dt \approx Q[f] = \sum_{i=0}^{q} w_i f(t_{m,i})$$

- Radau quadrature has algebraic degree $2q$
- $0 < w_i \leq \tau_m$

If $g \in P^q(I_m, L^2(0,1))$, then we can express the method:

$$Q[(U', v)] + Q[a(U, v)] + (\{U\}_{m-1}, v_+^{m-1}) = Q[(g, v)] \quad \forall v \in V_N^\tau$$

- Let $f \in C(0,1)$, we define Radau quadrature:

$$\int_{I_m} f(t)dt \approx Q[f] = \sum_{i=0}^{q} w_i f(t_{m,i})$$

- Radau quadrature has algebraic degree $2q$
- $0 < w_i \leq \tau_m$

If $g \in P^q(I_m, L^2(0,1))$, then we can express the method:

$$Q[(U', v)] + Q[a(U, v)] + (\{U\}_{m-1}, v_+^{m-1}) = Q[(g, v)] \quad \forall v \in V_N^\tau$$

For the exact solution we obtain:

$$Q[(u', v)] + Q[a(u, v)] + (\{u\}_{m-1}, v_+^{m-1}) = Q[(g, v)] \quad \forall v \in V_N^\tau$$

We divide the error

$$e(t) = U(t) - u(t) = \underbrace{U(t) - \pi u(t)}_{:=\xi(t)} + \underbrace{\pi u(t) - u(t)}_{:=\eta(t)}$$

Error estimates

We divide the error

$$e(t) = U(t) - u(t) = \underbrace{U(t) - \pi u(t)}_{:=\xi(t)} + \underbrace{\pi u(t) - u(t)}_{:=\eta(t)}$$

$$\int_{I_m} (\xi', v) + a(\xi, v)dt + (\{\xi\}_{m-1}, v_+^{m-1})$$
$$= -Q[(\eta', v)] - (\{\eta\}_{m-1}, v_+^{m-1}) - Q[a(\eta, v)].$$

Error estimates

$$
\begin{aligned}
Q[(\eta', v)] + (\{\eta\}_{m-1}, v_+^{m-1}) &\leq \tau_m C(\tau^{q+1} + (N^{-1} \ln N)^2) \sup_{I_m} \|v\| \\
Q[a(\eta, v)] &= 0
\end{aligned}
$$

Error estimates

$$
\begin{aligned}
Q[(\eta', v)] + (\{\eta\}_{m-1}, v_+^{m-1}) &\leq \tau_m C(\tau^{q+1} + (N^{-1} \ln N)^2) \sup_{I_m} \|v\| \\
Q[a(\eta, v)] &= 0
\end{aligned}
$$

$$
\int_{I_m} (\xi', v) + a(\xi, v) dt + (\{\xi\}_{m-1}, v_+^{m-1})
$$
$$
\leq \tau_m C(\tau^{q+1} + (N^{-1} \ln N)^2) \sup_{I_m} \|v\|.
$$

Error estimates

Setting $v = 2\xi$:

$$\|\xi_-^m\|^2 - \|\xi_-^{m-1}\|^2 + \|\{\xi\}_{m-1}\|^2 + 2\min(c_0, 1)\int_{I_m} \|\xi\|_\varepsilon^2 dt$$
$$\leq \tau_m C\left(\tau^{q+1} + (N^{-1}\ln N)^2\right) \sup_{I_m} \|\xi\|.$$

Estimates inside I_m

[Akrivis, Makridakis, 2004]: setting $v = 2\tilde{\xi} \in V_N^\tau$, where $\tilde{\xi}$ is interpolation over Radau quadrature nodes in I_m of $\frac{\tau_m}{t-t_{m-1}}\xi(t)$

Estimates inside I_m

[Akrivis, Makridakis, 2004]: setting $v = 2\tilde{\xi} \in V_N^\tau$, where $\tilde{\xi}$ is interpolation over Radau quadrature nodes in I_m of $\frac{\tau_m}{t-t_{m-1}}\xi(t)$ the values at the nodes of Radau quadrature:
$\tilde{\xi}(t_{m,i}) = \frac{\tau_m}{t_{m,i}-t_{m-1}}\xi(t_{m,i})$

Estimates inside I_m

[Akrivis, Makridakis, 2004]: setting $v = 2\tilde{\xi} \in V_N^\tau$, where $\tilde{\xi}$ is interpolation over Radau quadrature nodes in I_m of $\frac{\tau_m}{t-t_{m-1}}\xi(t)$ the values at the nodes of Radau quadrature:
$\tilde{\xi}(t_{m,i}) = \frac{\tau_m}{t_{m,i}-t_{m-1}}\xi(t_{m,i})$, where $\frac{\tau_m}{t_{m,i}-t_{m-1}} \geq 1$

Estimates inside I_m

[Akrivis, Makridakis, 2004]: setting $v = 2\tilde{\xi} \in V_N^\tau$, where $\tilde{\xi}$ is interpolation over Radau quadrature nodes in I_m of $\frac{\tau_m}{t-t_{m-1}}\xi(t)$ the values at the nodes of Radau quadrature:
$\tilde{\xi}(t_{m,i}) = \frac{\tau_m}{t_{m,i}-t_{m-1}}\xi(t_{m,i})$, where $\frac{\tau_m}{t_{m,i}-t_{m-1}} \geq 1$
Then

$$\int_{I_m} a(\xi, \tilde{\xi})dt = Q[a(\xi, \tilde{\xi})]$$

$$= \sum_i w_i \frac{\tau_m}{t_{m,i} - t_{m-1}} a(\xi(t_{m,i}), \xi(t_{m,i}))$$

$$\geq \min(c_0, 1)Q[\|\xi\|_\varepsilon^2] = \min(c_0, 1)\int_{I_m} \|\xi\|_\varepsilon^2 dt$$

Lemma

Let $v \in V_N^\tau$ and $\tilde{v}$ be the interpolation of $\frac{\tau_m}{t - t_{m-1}} v(t)$ over Radau quadrature nodes in I_m. Then

$$2 \int_{I_m} (v', \tilde{v}) dt + 2(v_+^{m-1}, \tilde{v}_+^{m-1}) = \|v_-^m\|^2 + \frac{1}{\tau_m} \int_{I_m} \|\tilde{v}\|^2 dt \qquad (3)$$

Estimates inside I_m

Since the norms $\sup_{I_m} \|\xi\|^2$, $\frac{1}{\tau_m} \int_{I_m} \|\tilde{\xi}\|^2 dt$ and $\sup_{I_m} \|\tilde{\xi}\|^2$ are equivalent, we get

$$\sup_{I_m} \|\xi\|^2 \leq C \frac{1}{\tau_m} \int_{I_m} \|\tilde{\xi}\|^2 dt$$

$$\leq C \left(\|\xi_-^m\|^2 + \frac{1}{\tau_m} \int_{I_m} \|\tilde{\xi}\|^2 dt + 2 \min(c_0, 1) \int_{I_m} \|\xi\|_\varepsilon^2 dt \right)$$

$$\leq C \left((\xi_-^{m-1}, \tilde{\xi}_+^{m-1}) + C \left(\tau^{q+1} + (N^{-1} \ln N)^2 \right) \sup_{I_m} \|\xi\| \right)$$

$$\leq C \left(\|\xi_-^{m-1}\|^2 + \tau^{2q+2} + (N^{-1} \ln N)^4 \right) + \frac{1}{2} \sup_{I_m} \|\xi\|^2$$

Error estimates

Theorem

Let u be exact solution of singularly perturbed problem (1) and U be its discrete approximation. Then

$$\max_{m=1,\dots,r} \sup_{I_m} \|U - u\| \leq C((N^{-1} \ln N)^2 + \tau^{q+1}).$$

Thank you for your attention.