Analysis of pattern formation using numerical continuation

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Outline

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Problem

Consider reaction-diffussion system for two species **u** and **v** in 1D domain $x \in [0, \ell]$,

$$\begin{aligned} \mathbf{u}_t &= d_1\,\mathbf{u}_{xx} + \bar{f}(\mathbf{u},\mathbf{v}) \\ \mathbf{v}_t &= d_2\,\mathbf{v}_{xx} + \bar{g}(\mathbf{u},\mathbf{v}) \end{aligned}$$

Context: A theory of biological pattern formation.
Self-organization in development biology, morphogenesis

Objective: domain size driven instability

V. Klika, M. Kozák and E.A. Gaffney: *Domain Size Driven Instability: Self-Organization in Systems with Advection*, SIAM J. Appl. Math., 2018, pp 2298–2322.

Domain can be scalled to the unit interval $0 \le x \le 1$ introducing parameter L. Hence, we consider

$$\begin{aligned} \mathbf{u}_t &= \frac{d_1}{L^2}\mathbf{u}_{xx} + f(\mathbf{u},\mathbf{v}) \\ \mathbf{v}_t &= \frac{d_2}{L^2}\mathbf{v}_{xx} + g(\mathbf{u},\mathbf{v}) \end{aligned}$$

in domain $0 \le x \le 1$. Here L is the length of the interval. We consider Neumann boundary conditions (zero flux)

$$\mathbf{u}_x(0,t) = \mathbf{u}_x(1,t) = 0\,, \quad \mathbf{v}_x(0,t) = \mathbf{v}_x(1,t) = 0\,.$$

We seek for *steady states*.

Homogeneous steady state: There exists $u^* \in \mathbb{R}^1$, $v^* \in \mathbb{R}^1$, such that

$$f(\mathbf{u}(x,0),\mathbf{v}(x,0)) = f(u^*,v^*) = g(\mathbf{u}(x,0),\mathbf{v}(x,0)) = g(u^*,v^*) = 0\,,\ 0 \le x \le 1\,.$$

Note that, in this case, $\mathbf{u}_{xx} = 0$ and $\mathbf{v}_{xx} = 0$ in the domain $0 \le x \le 1$.

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Discretization of the model

Method of lines: Define the equidistant mesh on the interval $0 \le x \le 1$

$$x_j = jh\,, \quad h = \frac{1}{N+1}\,, \quad j = 1, \ldots, N\,, \qquad (2.1)$$

where N is the number of meshpoints.
The state variables **u**, **v** are approximated by discrete state variables

$$\mathbf{u} \approx [u_1, \ldots, u_i, \ldots, u_N]^T \in \mathbb{R}^N\,, \quad \mathbf{v} \approx [v_1, \ldots, v_i, \ldots, v_N]^T \in \mathbb{R}^N,$$

We seek for discrete steady states. They depend on the parameter L^2.

We define

$$F \; : \; \mathbb{R}^{2N} \times \mathbb{R}^1 \longrightarrow \mathbb{R}^{2N} \tag{2.2}$$

and seek for the roots

$$F(w, L^2) = 0\,, \quad w \in \mathbb{R}^{2N}\,, \quad w_i = u_i\,, \quad w_{N+i} = v_i\,, \quad i = 1, \ldots, N\,.$$

This set is called *solution manifold*.

We assume the existence of *homogeneous steady state*

$$F(w^*, L^2) = 0\,, \quad w^* \in \mathbb{R}^{2N}\,, \quad w_i^* = u^*\,, \quad w_{N+i}^* = v^*\,, \quad i = 1, \ldots, N\,.$$

Primary bifurcation point

Definition. Consider the particular homogeneous steady state $w^* \in \mathbb{R}^{2N}$, $(L^*)^2 \in \mathbb{R}^1$,

$$F(w^*, (L^*)^2) = 0 \in \mathbb{R}^{2N}, \quad A \equiv F_w(w^*, (L^*)^2), \quad \dim \mathrm{Ker}\, A = 1 .$$

Let ξ and η be right and left eigenvectors corresponding to the zero eigenvalue

$$A\xi = 0 \in \mathbb{R}^{2N}, \quad \|\xi\| = 1, \quad A^T\eta = 0 \in \mathbb{R}^{2N}, \quad \|\eta\| = 1, \quad \eta^T\xi \neq 0,$$

with an algebraic multiplicity equal to one. Then the point $(w^*, (L^*)^2) \in \mathbb{R}^{2N+1}$ is called *primary bifurcation point*.

... Lyapunov-Schmidt reduction, bifurcation equation

M. Golubitsky, I. Stewart and D.G. Schaeffer: *Singularities and groups in bifurcation theory, Vol II*, Springer, 1988

Example. Consider the Schnackenberg model for the parameter setting $a = 0.1$, $b = 0.9$, $\gamma = 10$, $d_1 = 0.1$, $d_2 = 1.6$ and $N = 20$.
Homogeneous state state w^*: $u^* = a + b = 1$, $v^* = b/(a+b)^2 = 0.9$.
The aim is to compute the branch of inhomogeneous steady states emanating from a particular primary bifurcation point.

The primary bifurcation point $w^*, (L^*)^2$ with least $(L^*)^2 > 0$ is $(L^*)^2 = 0.153969537228066$.

J. Schnakenberg: *Simple chemical reaction systems with limit cycle behaviour*, J. Theoret. Biol., Vol 81, 1979, pp 389–400.

Solution manifold via numerical continuation

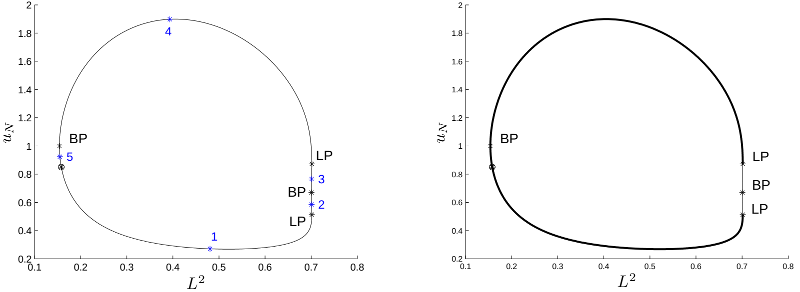

Figure: On the left: Branch of inhomogeneous steady states projected onto pairs (u_N, L^2). It is oriented anticlockwise. The branch emanates from the primary bifurcation point $u_N = 1$, $(L^*)^2 \approx 0.1539$ labeled as BP. The circle on the branch marks the initial point of continuation procedure. In points $1, \ldots, 5$ we test stability. The indicated bifurcation points are (in clockwise directions) LP, secondary bifurcation point BP, LP and the primary bifurcation point BP. On the right: thick and thin segments of curves indicate stable and unstable steady states.

A. Dhooge, W. Govaerts and Yu.A. Kuznetsov: *MATCONT*: a MATLAB package for numerical bifurcation analysis of ODEs, ACM Transactions on Mathematical Software Vol 29, No.2., 2003, pp 141–164.

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Dispersion equation

Let $u^* \in \mathbb{R}^1$, $v^* \in \mathbb{R}^1$ be a homogeneous steady steady i.e.,

$$f(u^*, v^*) = g(u^*, v^*) = 0\,,\; 0 \leq x \leq 1\,.$$

In particular, in the model Schnakenberg model,

$$u^* = a + b\,, \quad v^* = b/(a+b)^2\,, \quad a > 0, b > 0\,.$$

We define Jacobian at the steady state $u^* \in \mathbb{R}^1$, $v^* \in \mathbb{R}^1$:

$$\mathbf{J} = \left[\begin{array}{cc} f_u & f_v \\ g_u & g_v \end{array} \right]_{u,v:=u^*,v^*} \,.$$

In particular, in the model Schnakenberg model,

$$\mathbf{J} = \left[\begin{array}{cc} 2\gamma b/(a+b) - 1 & (a+b)^2 \\ -2b/(a+b) & -(a+b)^2 \end{array} \right], \quad a > 0, b > 0, \gamma > 0\,,$$

Let d_1 end d_2 be diffusion parameters. Let **J** be Jacobian corresponding to a steady state. The equation

$$\det\left(\mathbf{J} - k^2 \begin{bmatrix} d_1 & 0 \\ 0 & d_1 \end{bmatrix} - \lambda \mathbf{I}\right) = 0\,, \quad \mathbf{I} = \mathbf{I}_{2\times 2} \in \mathbb{R}^{2\times 2}$$

is called *dispersion relation*. It depends on *wavenumber* k^2 and *frequency* λ. Dispersion relation implicitly defines the relationship $\lambda = \lambda(k^2)$.

Instead of analyzing roots of dispersion relation we analyse the spectrum of the following matrix

$$\mathbf{H} \equiv \mathbf{J} - k^2 \begin{bmatrix} d_1 & 0 \\ 0 & d_1 \end{bmatrix}\,.$$

The spectrum $\sigma(\mathbf{H})$ consists of two eigenvalues $\{\lambda_1(k^2), \lambda_2(k^2)\}$. These can be numerically computed as a function of k^2. We use Matlab function `eig(H)`. Note that generally $\sigma(\mathbf{H})$ consists of either two real eigenvalues or a complex congate pair.

Given k^2, we define

$$\Re\left(\lambda_*(k^2)\right) = \max\left\{\Re\left(\lambda_i(k^2)\right)\right\}_{i=1,2}$$

which is the right-most eigenvalue of $\sigma(\mathbf{H})$. We call $\Re\left(\lambda_*(k^2)\right)$ *maximal growth rate*.

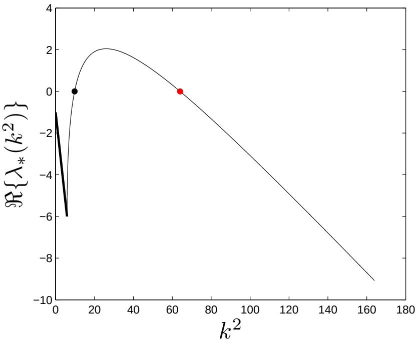


Figure: Maximal growth rate $k^2 \mapsto \Re\left(\lambda_*(k^2)\right)$: thin and thick curve segments refers to two real eigenvalues and complex conjugate pairs, respectively. Case: Schnakenberg model, $a = 0.1$, $b = 0.9$, $\gamma = 10$, $d_1 = 0.1$, $d_2 = 1.6$.

Let k_-^2 and k_+^2, $k_-^2 \le k_+^2$, be the roots of function $k^2 \mapsto \Re\left(\lambda_*(k^2)\right)$. We can get the closed-form expressions

$$k_-^2 = \frac{d_2 f_u + d_1 g_v - \sqrt{\Delta}}{2 d_1 d_2}, \quad k_+^2 = \frac{d_2 f_u + d_1 g_v + \sqrt{\Delta}}{2 d_1 d_2}, \tag{3.3}$$

where $\Delta = (d_2 f_u + d_1 g_v)^2 - 4 d_1 d_2 \det \mathbf{J}$, and $\mathbf{J}$, f_u, g_v are related to Jacobian.

The open interval $\left(k_-^2, k_+^2\right)$ is called *range of growing wavenumbers*. For a given k^2, $k_-^2 < k^2 < k_+^2$, the maximal growth rate is positive.

Spatial pattern: linear stability analysis

Linearization about a homogeneous steady state $u^* \in \mathbb{R}^1$, $v^* \in \mathbb{R}^1$:

$$\begin{bmatrix} \mathbf{u}_t \\ \mathbf{v}_t \end{bmatrix} = \mathbf{J} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} + \frac{1}{L^2} \begin{bmatrix} d_1 & 0 \\ 0 & d_1 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{xx} \\ \mathbf{v}_{xx} \end{bmatrix} \tag{3.4}$$

in domain $0 \le x \le 1$. We employ Fourier analysis, setting

$$\begin{bmatrix} \mathbf{u}(x,t) \\ \mathbf{v}(x,t) \end{bmatrix} = \sum_{j=1}^{+\infty} \mathbf{c}_j e^{\lambda t} W_j(x) , \quad 0 \le x \le 1 , \, t \ge 0 , \tag{3.5}$$

where $\mathbf{c}_j \in \mathbb{R}^2$ are coefficients to be determined, $\lambda \in \mathbb{R}^1$ defines temporal growth.

J.D. Murray: *Mathematical biology. II*, Springer-Verlag, New York, 2003.

The Fourier functions W_j are related to $1-D$ Laplacians with Neumann boundary conditions

$$W_j'' + k^2 W_j = 0\,, \quad 0 \le x \le 1\,, \quad W_j'(0) = W_j'(1) = 0\,. \tag{3.6}$$

The relevant eigenvalues and eigenvectors are $k^2 = (j\pi)^2$ and $W_j = \sqrt{2}\cos(j\pi x)$, for $j = 1, 2, \ldots$.

We conclude that

$$\det\left(\mathbf{J} - \frac{k^2}{L^2}\begin{bmatrix} d_1 & 0 \\ 0 & d_1 \end{bmatrix} - \lambda \mathbf{I}\right) = 0\,, \quad \mathbf{I} = \mathbf{I}_{2\times 2} \in \mathbb{R}^{2\times 2}\,. \tag{3.7}$$

This equation is analogous to the dispersion relation. It depends on frequency λ, particular eigenvalue $k^2 = (j\pi)^2$, $j = 1, 2, \ldots$ and wavelength L^2.

The roots of the above dispersion relation are related to the spectrum of the matrix

$$\mathbf{H} \equiv \mathbf{J} - \frac{k^2}{L^2} \begin{bmatrix} d_1 & 0 \\ 0 & d_1 \end{bmatrix} . \tag{3.8}$$

Its spectrum $\sigma(\mathbf{H})$ consists of two eigenvalues $\left\{\lambda_1\left(\frac{k^2}{L^2}\right), \lambda_2\left(\frac{k^2}{L^2}\right)\right\}$.
We define the maximal growth rate as function

$$\frac{k^2}{L^2} \mapsto \Re\left(\lambda_*\left(\frac{k^2}{L^2}\right)\right) \equiv \max\left\{\Re\left(\lambda_i\left(\frac{k^2}{L^2}\right)\right)\right\}_{i=1,2} . \tag{3.9}$$

We ask for $\frac{k^2}{L^2}$ when $\Re\left(\lambda_*\left(\frac{k^2}{L^2}\right)\right)$ is positive.
We observe that for parameters $k^2 = (j\pi)^2$ and L^2 which satisfy

$$k_-^2 < \frac{k^2}{L^2} < k_+^2$$

we can expect the existence of an inhomogeneous steady state.

In terms of wavelength L^2, we arrive at the restriction

$$\frac{k^2}{k_+^2} < L^2 < \frac{k^2}{k_-^2}\,, \quad k^2 = (j\pi)^2\,. \tag{3.10}$$

The estimate (3.10) of the stable wavelength range is related to the partial differential equation (3.4). The state variables are functions $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$, see (3.5).

However, when it comes to the discrete model (2.2) of reality, we must consistently consider Laplacian on an equidistant grid (2.1):

Definition.

$[e\_val, e\_vec] = Lap\_neum(N, No)$

$e\_val = \frac{2}{h^2}\left[1 - \cos\left(\frac{j\pi}{N}\right)\right]\,, \quad No = j \in \{1, \ldots, N-1\}\,, \quad h = \frac{1}{N+1}$,

$e\_vec \in \mathbb{R}^N$, the sampled eigenfunction of W_j, see (3.6).

Discrete Laplacian on an equidistant grid

Properties. $[e\_val, e\_vec] = Lap\_neum(N, No)$

- N is number of meshpoints
- No is serial number of eigenmode, $No = j \in \{1, \ldots, N-1\}$

The function *Lap_neum* returns corresponding eigenvalue $e\_val \in \mathbb{R}^1$ and eigenvector $e\_vec \in \mathbb{R}^N$. By definition: If $No = 0$ then $e\_val = 0$. Positive eigenvalues are arranged: $No \in \{1, \ldots, N-1\}$ from the smallest to the largest.
We formulate appropriate discrete analogies of (3.7), (3.8), (3.9) and (3.10). In particular, we set

$$k^2 = e\_val = Lap\_neum(N, No)\,, \quad No = j \in \{1, \ldots, N-1\} \tag{3.11}$$

In order to simplify notation, *e_val* and *e_vec* are generic value variables.

Critical wavelengths

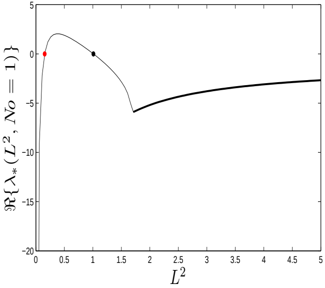
Definition. Given a mode number $No = j \in \{1, \ldots, N-1\}$, we define interval

$$\frac{e\_val}{k_+^2} < L^2 < \frac{e\_val}{k_-^2} \tag{3.12}$$

and call it *stable wavelength range*. We define

$$L2\_No\_up = \frac{e\_val}{k_+^2}, \quad L2\_No\_down = \frac{e\_val}{k_-^2} \tag{3.13}$$

as *critical wavelengths* related to mode number $No = j \in \{1, \ldots, N-1\}$.



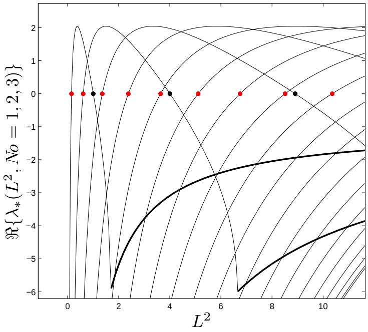


Figure: Critical wavelengths, see Example: On the left: *L2_1_up* $\approx$ 0.1542, *L2_1_down* $\approx$ 1.0098. On the right: sorted in ascending order: *L2_1_up*, *L2_2_up*, *L2_1_down*, *L2_3_up*, *L2_4_up*, *L2_5_up*, *L2_2_down*, *L2_6_up*, etc.

Critical wavelengths: primary bifurcation

Theorem. For a given $j \in \{1, \dots, N-1\}$, let $L2\_j\_up$ and $L2\_j\_down$ be the critical wavelengths. Then $L2\_j\_up$ and $L2\_j\_down$ are connected to the following primary bifurcation points:

- $F(w^*, L2\_j\_up) = 0 \in \mathbb{R}^N$,
  $$F_w(w^*, L2\_j\_up) \begin{bmatrix} a_{up} \\ b_{up} \end{bmatrix} \otimes e\_vec \in \mathbb{R}^{2N} = 0 \in \mathbb{R}^{2N}$$
- $F(w^*, L2\_j\_down) = 0 \in \mathbb{R}^N$,
  $$F_w(w^*, L2\_j\_down) \begin{bmatrix} a_{down} \\ b_{down} \end{bmatrix} \otimes e\_vec \in \mathbb{R}^{2N} = 0 \in \mathbb{R}^{2N}$$

The coefficients $a_{up} \in \mathbb{R}^1$, $b_{up} \in \mathbb{R}^1$ and $a_{down} \in \mathbb{R}^1$, $b_{down} \in \mathbb{R}^1$ are defined by a formula.

Therefore, we can identify the critical wavelength with the primary bifurcation point. We can compute and sort all $2(N-1)$ bifurcation points.

Outline

Matrix representation

Consider abstract group $\Gamma = \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{\iota, \kappa_1, \kappa_2, \kappa_1\kappa_2\}$, $\kappa_1\kappa_2 = \kappa_2\kappa_1$.
Here

- $\mathbb{Z}_2$ is a cyclic group of order 2
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is a direct sum of groups.

Γ is therefore an Abelian group.

<u>Preliminaries</u>
We consider the following matrices:
identity matrix end *zero matrix* and *exchange matrix*
of the proper size:

$$\mathbf{I}_{2N\times 2N} \in \mathbb{R}^{2N\times 2N}, \quad \mathbf{O}_{N\times N} \in \mathbb{R}^{N\times N}, \quad \mathbf{E} = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \in \mathbb{R}^{N\times N}$$

We define matrix representation of the group Γ in the state space $\mathbb{R}^{2N}$:

$$\mathbf{G}(\iota) = \mathbf{I}_{2N\times 2N} \in \mathbb{R}^{2N\times 2N}\,, \quad \mathbf{G}(\kappa_1) = -\mathbf{I}_{2N\times 2N} \in \mathbb{R}^{2N\times 2N}\,,$$

$$\mathbf{G}(\kappa_2) = \begin{bmatrix} \mathbf{E} & \mathbf{O}_{N\times N} \\ \mathbf{O}_{N\times N} & \mathbf{E} \end{bmatrix} \in \mathbb{R}^{2N\times 2N}\,,$$

$$\mathbf{G}(\kappa_1\kappa_2) = \mathbf{G}(\kappa_1)\mathbf{G}(\kappa_2) = -\mathbf{G}(\kappa_2) = \mathbf{G}(\kappa_2\kappa_1)\,.$$

We conclude, that $\Gamma = \{\mathbf{G}(\iota), \mathbf{G}(\kappa_1), \mathbf{G}(\kappa_2), \mathbf{G}(\kappa_1\kappa_2)\}$ is Abelian group with faithful representation on state space $\mathbb{R}^{2N}$. Each group element $\gamma \in \Gamma$ is identified with its action namely a linear transformation on $\mathbb{R}^{2N}$.

In order to simplify notation, we identify the mentioned linear transformations with group elements $\Gamma = \{\iota, \kappa_1, \kappa_2, \kappa_1\kappa_2\}$.

Maximal isotropy subgroups

The group Γ has proper subgroups
$\Sigma_{\kappa_1} = \{\iota, \kappa_1\}$, $\Sigma_{\kappa_2} = \{\iota, \kappa_2\}$, $\Sigma_{\kappa_1\kappa_2} = \{\iota, \kappa_1\kappa_2\}$ and $\Sigma_0 = \{\iota\}$.
We need to recall two notions of representation theory:

- *the isotropy subgroup* Σ
- *the fixed point subspace* $\mathrm{Fix}\,\Sigma$ *of a subgroup* Σ.

It holds

$$\begin{cases} \mathrm{Fix}_{\mathbb{R}^{2N}}\,\Sigma_{\kappa_1} &= \mathrm{Fix}_{\mathbb{R}^{2N}}\,\Gamma = 0 \in \mathbb{R}^{2N}\,, \\ \mathrm{Fix}_{\mathbb{R}^{2N}}\,\Sigma_{\kappa_2} &= \{w \in \mathbb{R}^{2N} : w_i = w_{N-i+1}\,,\, w_{N+i} = w_{2N-i+1}\,, \\ & \qquad i = 1, \ldots, N\}\,, \\ \mathrm{Fix}_{\mathbb{R}^{2N}}\,\Sigma_{\kappa_1\kappa_2} &= \{w \in \mathbb{R}^{2N} : w_i = -w_{N-i+1}\,,\, w_{N+i} = -w_{2N-i+1}\,, \\ & \qquad i = 1, \ldots, N\}\,, \\ \mathrm{Fix}_{\mathbb{R}^{2N}}\,\Sigma_0 &= \mathbb{R}^{2N}\,. \end{cases}$$

Σ_{κ_2}, $\Sigma_{\kappa_1\kappa_2}$ and Σ_0 are isotropy subgroups of Γ. Moreover, Σ_{κ_2} and $\Sigma_{\kappa_1\kappa_2}$ are *maximal isotropy subgroups*.

The symmetry of the solution: how it manifests itself

A key feature of Schnackenberg's model is its Γ*-equivariance*:

$$F(\gamma w, L^2) = \gamma F(w, L^2)$$

for $(w, L^2) \in \mathbb{R}^{2N} \times \mathbb{R}^1$, for all $\gamma \in \{\iota, \kappa_1, \kappa_2, \kappa_1\kappa_2\}$.

Consider Schnakenberg model, see Example, assuming $N = 40$:

From the homogeneous steady state bifurcate the following inhomogeneous steady states (in ascending order)
L2_1_up, *L2_2_up*, *L2_1_down*, *L2_3_up*, *L2_4_up*, *L2_5_up*, *L2_2_down*, *L2_6_up*, etc.

Consider the branches emanating from *L2_3_up* and *L2_4_up*. We ask for a comparison.

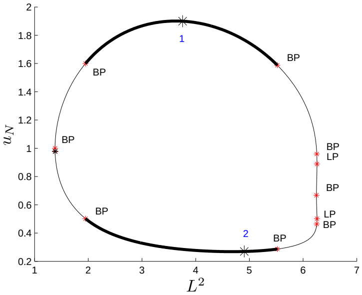
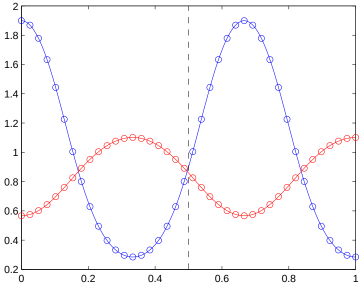


Figure: Schnakenberg model, see Example, $N = 40$. Branches of inhomogeneous steady states projected onto pairs (u_N, L^2), emanating from *L2_3_up* (on the left) and *L2_4_up* (on the right). Thick and thin segments of curves indicate stable and unstable steady states. Placement of random test points $*$ on stable segments.

$\Sigma_{\kappa_1\kappa_2}$ discrete *odd functions* = *antisymmetric*



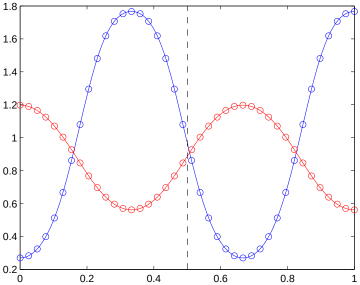


Figure: Schnakenberg model, see Example, $N = 40$, the branch $L2\_3\_up$: Consider the steady states $[u_1, \ldots, u_N, v_1, \ldots, v_N] \in \mathbb{R}^{2N}$ related the two marked test points $*$ in the branch $L2\_3\_up$. Both steady states $[u_1, \ldots, u_N]$ (in blue), and $[v_1, \ldots, v_N]$ (in red) have rotational symmetry with respect to the center line $x = 1/2$.

activator/inhibitor = red/blue

Σ_{κ_2} discrete *even functions = symmetric*

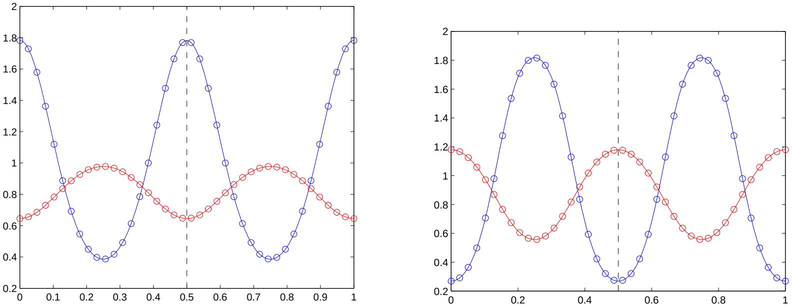


Figure: Schnakenberg model, see Example, $N = 40$, the branch *L2_4_up*: Consider the steady states $[u_1, \ldots, u_N, v_1, \ldots, v_N] \in \mathbb{R}^{2N}$ related the two marked test points $*$ in the branch *L2_4_up*. Both steady states $[u_1, \ldots, u_N]$ (in blue), and $[v_1, \ldots, v_N]$ (in red) have symmetry of reflection with respect to the center line $x = 1/2$.

activator/inhibitor = red/blue

$$L^2, \quad \mathtt{asym} \equiv \frac{u_1 - u_N}{2}, \quad \mathtt{sym} \equiv \frac{u_1 + u_N}{2}.$$

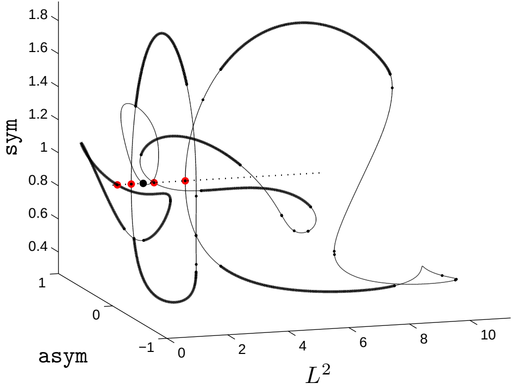


Figure: Schnakenberg model, see Example. Homogeneous steady states (dashed). Branches of inhomogeneous steady states emanating from *L2_1_up*, *L2_2_up*, *L2_1_down*, *L2_3_up*, *L2_4_up*.

Outline

Conclusions

Algorithmic contribution to the continuation algorithm: a) initialization of the branch (via Taylor expansion of the bifurcation equation), b) segmenting the branch, distinquishing stable and unstable branch segments.

Results

- classification of model symmetries
- construction of a global bifurcation diagram

Outline

Supplement: Dynamic simulation

Recall

$$F \; : \; \mathbb{R}^{2N} \times \mathbb{R}^1 \longrightarrow \mathbb{R}^{2N}$$

$$F(w, L^2) = 0\,, \quad w \in \mathbb{R}^{2N}\,, \quad w_i = u_i\,, \quad w_{N+i} = v_i\,, \quad i = 1, \ldots, N\,.$$

Dynamic simulation. Define the initial value problem

$$w'(t) = F(w(t), L^2)\,,$$

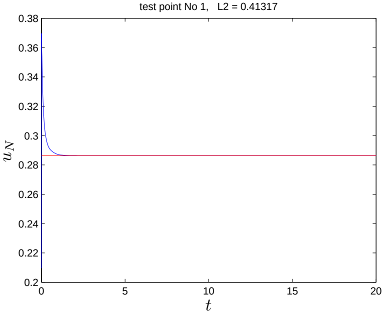
$$t \in \mathbb{R}^1 \longmapsto w(t) = [u_1(t), \ldots, u_N(t), v_1(t), \ldots, v_N(t)]^T \in \mathbb{R}^{2N}\,.$$

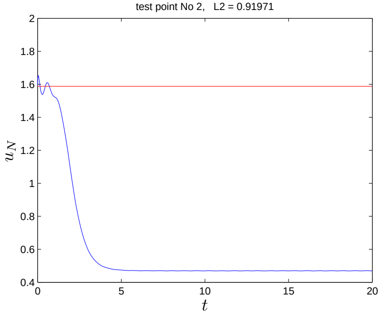
L is fixed. The initial condition $w^0 \in \mathbb{R}^{2N}$ is usually considered to be randomly perturbed *homogeneous steady state*.

It is expected that $w(t) \in \mathbb{R}^{2N} \longmapsto [u_1, \ldots, u_N, v_1, \ldots, v_N]^T \in \mathbb{R}^{2N}$ as $t \in \mathbb{R}^1 \longmapsto \infty$. Thus, in the case of convergence, the vector $[u_1, \ldots, u_N, v_1, \ldots, v_N]^T$ can be an *inhomogeneous* steady state related to the selected parameter L.

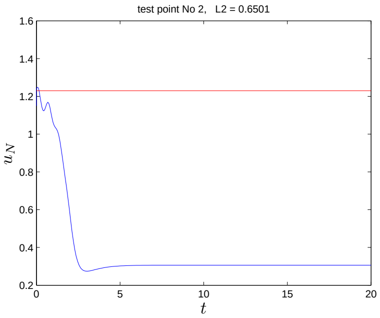
We are going to illustrate that

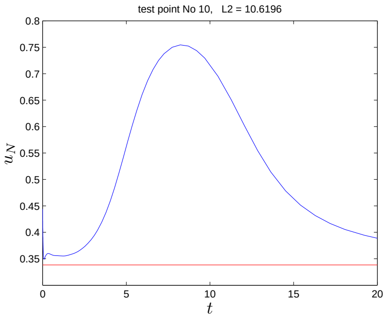
1. dynamic simulations can converge to steady states that are not unique,
2. it may not converge at all.
3. in principle, the dynamic simulation can only provide stable steady states.



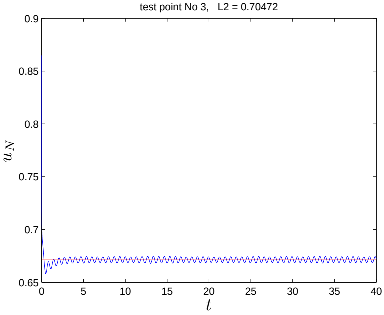


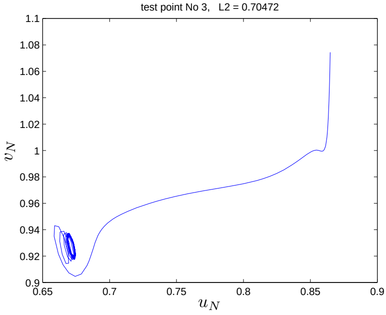
On the left: Consider the branch *L*2_1_*up*. Dynamic simulation: random perturbation at stable test point No 1. Steady state (red), resulting trajectory (blue).
On the right: Consider the branch *L*2_1_*down*. Dynamic simulation: random perturbation at an unstable test point No 2. Steady state (red), resulting trajectory (blue).





On the left: Consider the branch *L2_2_up*. Dynamic simulation: random perturbation at unstable test point No 2. Steady state (red), resulting trajectory (blue).
On the right: Consider the branch *L2_5_up*. Dynamic simulation: random perturbation at stable test point No 10. Steady state (red), resulting trajectory (blue). Observe the *Transient growth*.





The branch *L*2_1_*up*. Dynamic simulation: random perturbation at unstable test point No 3.
On the left: Steady state (red), resulting oscillation (blue).
On the right: Phase portrait of the oscillations.