The multiplicative Schwarz method for matrices with a special block structure

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joint work with

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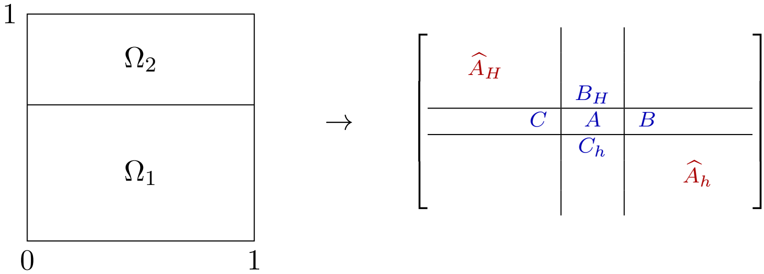
Motivation

Discretization of PDE's using finite differences



How to solve efficiently the corresponding linear systems?

Related papers

- **C. Echeverría, J. Liesen, P. Tichý, D. Szyld**, Convergence of the multiplicative Schwarz method for singularly perturbed convection-diffusion problems discretized on a Shishkin mesh, Electron. Trans. Numer. Anal. (ETNA), 2018.
- **C. Echeverría, J. Liesen, and P. Tichý**, Analysis of the multiplicative Schwarz method for matrices with a special block structure, ETNA, 2021.

Solving linear systems with the system matrix

$$
\left[\begin{array}{c|c|c}
\color{red}{\widehat{A}_H} & \color{blue}{B_H} & \\
\hline
\color{blue}{C} & \color{blue}{A} & \color{blue}{B} \\
\hline
 & \color{blue}{C_h} & \color{red}{\widehat{A}_h}
\end{array}\right]
$$

Linear solver

and geometry of the problem

$$\mathcal{A} = \left[\begin{array}{c|c|c} \widehat{A}_H & & \\ & B_H & \\ \hline & C \quad A \quad B & \\ \hline & C_h & \\ & & \widehat{A}_h \end{array}\right] \in \mathbb{R}^{N(2m+1)\times N(2m+1)}$$

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- Systems with submatrices easily solvable (Toeplitz).
- Use the **multiplicative Schwarz method**.
- Restriction op. $R_1 = \left[\begin{array}{cc} I_{N(m+1)} & 0 \end{array}\right]$, $R_2 = \left[\begin{array}{cc} 0 & I_{N(m+1)} \end{array}\right]$.

Multiplicative Schwarz method

- Given $x^{(k)}$, then $x = x^{(k)} + y$ and y satisfies

$$\mathcal{A}y = b - \mathcal{A}x^{(k)} \equiv r^{(k)}.$$

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- **Restriction** to the **first domain**

$$(R_1 \mathcal{A} R_1^T)\, y_1 \ = \ R_1 r^{(k)}$$

and **prolongation**

$$x^{(k+\frac{1}{2})} = x^{(k)} + R_1^T y_1.$$

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$$x^{(k+\frac{1}{2})} = x^{(k)} + R_1^T y_1.$$

- Do the same with $x^{(k+\frac{1}{2})}$ on the **second domain**

$$x^{(k+1)} = x^{(k+\frac{1}{2})} + R_2^T y_2.$$

Multiplicative Schwarz method

as a stationary method

- Consistent stationary method

$$x^{(k)} \quad = \quad \overbrace{(I - P_2)(I - P_1)}^{T} \, x^{(k-1)} + v$$

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- Bounds

$$\|x - x^{(k)}\| \leq \|T^k\| \, \|x - x^{(0)}\| \leq \|T\|^k \, \|x - x^{(0)}\|$$

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$$\|x - x^{(k)}\| \; \leq \; \|T^k\| \, \|x - x^{(0)}\| \; \leq \; \|T\|^k \, \|x - x^{(0)}\|$$

- We would like to show that

$$\|T^k\| = \rho^k, \qquad \rho \; \leq ?$$

Schwarz method as a preconditioner

- Consistent scheme

$$x^{(k+1)} = T\, x^{(k)} + v$$

- **Preconditioned** system

$$(I - T)x = v$$

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- If T has **rank** ℓ, then

$$\dim\left(\mathcal{K}_k(I - T, r_0)\right) \leq \ell + 1$$

and GMRES converges in **at most** $\ell + 1$ **iterations**.

Structure of T for

$$
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$$

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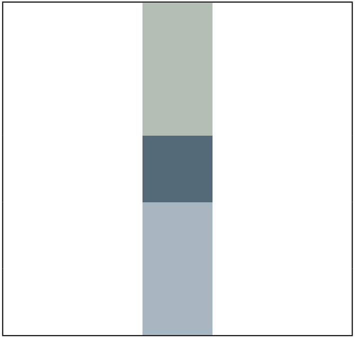
Structure of T



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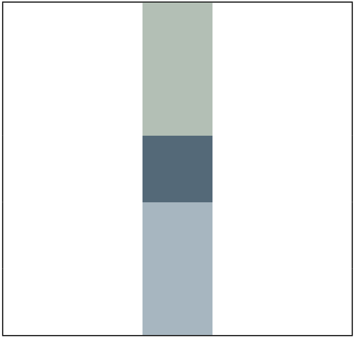

$$T = (I - P_2)(I - P_1)$$

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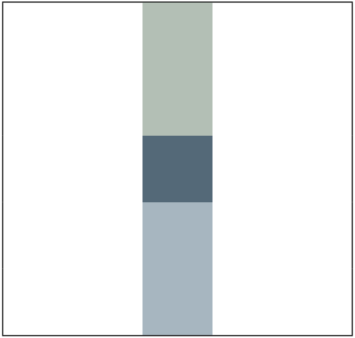


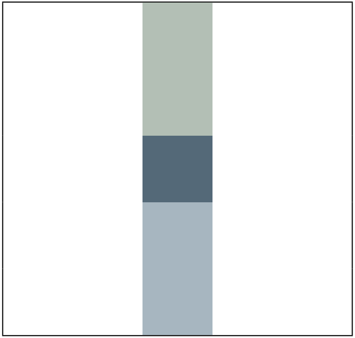
$$T = (I - P_2)(I - P_1)$$

$$\|T^{k+1}\| \;\leq\; \rho^k \, \|T\|$$

$$\rho = \|Z_{11}^{(h)} C_h \Pi^{(2)} Z_{mm}^{(H)} B_H \Pi^{(1)}\|$$

$$\widehat{A}_h^{-1} = [Z_{ij}^{(h)}], \quad \Pi^{(2)} = \left(A - B Z_{11}^{(h)} C_h\right)^{-1} C$$

Structure of T



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How to bound **norms of blocks** of inverses of $\widehat{A}_h$ and $\widehat{A}_H$?

Block tridiagonal case

$$\left[\begin{array}{c|c|c} \widehat{A}_H & & \\ & B_H & \\ \hline C & A & B \\ \hline & C_h & \\ & & \widehat{A}_h \end{array}\right]$$

Block tridiagonal case

New results of [Echeverría, Liesen, Nabben, 2018]

$$\widehat{A}_h = \begin{bmatrix} A_h & B_h & & \\ C_h & \ddots & \ddots & \\ & \ddots & \ddots & B_h \\ & & C_h & A_h \end{bmatrix}, \qquad \widehat{A}_H = \dots$$

$\widehat{A}_h$ is **row block diagonally dominant** if

$$\|A_h^{-1} B_h\| + \|A_h^{-1} C_h\| \leq 1$$

How to bound $\|Z_{ij}^{(h)}\|$? [Echeverría, Liesen, Nabben, 2018]

Bounding ρ

for $\mathcal{A}$ **row and column** block **diagonally dominant**

Using [Echeverría, Liesen, Nabben, 2018] we have shown

$$\rho \;\leq\; \frac{\eta_h \|A^{-1}C\|}{1-\eta_h\|A^{-1}B\|}\,\frac{\eta_H \|A^{-1}B\|}{1-\eta_H\|A^{-1}C\|}$$

where $\|\cdot\|$ is any induced matrix norm and

$$\eta_h \;=\; \min\left\{\frac{\|A_h^{-1}C_h\|}{1-\|A_h^{-1}B_h\|}, \frac{\|A_h^{-1}\|\|C_h\|}{1-\|C_hA_h^{-1}\|}\right\}$$

Bounds now contain only inverses of **individual blocks**.

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Bounds now contain only inverses of **individual blocks**.

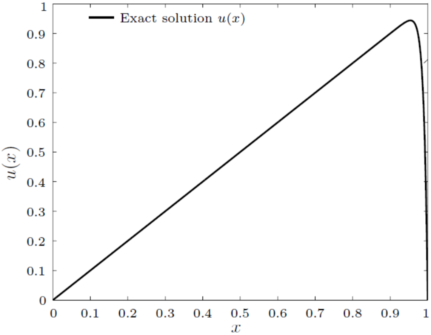
$$\|x-x^{(k)}\| \;\le\; \rho^k\,\|T\|\,\|x-x^{(0)}\|\,.$$

Application to singularly perturbed convection-diffusion equation

$$-\varepsilon \Delta u + \alpha u_y + \beta u = f$$

One-dimensional case

$$-\varepsilon\, u'' + \alpha\, u' + \beta\, u = f$$



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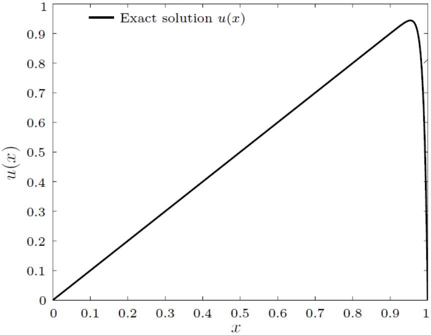
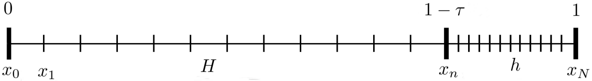


Shishkin mesh → uniform convergence



The standard upwind difference scheme

$$\mathcal{A} = \left[\begin{array}{c|c|c} A_H & & \\ & b_H & \\ \hline & c \quad a & b \\ \hline & c_h & \\ & & A_h \end{array}\right]$$

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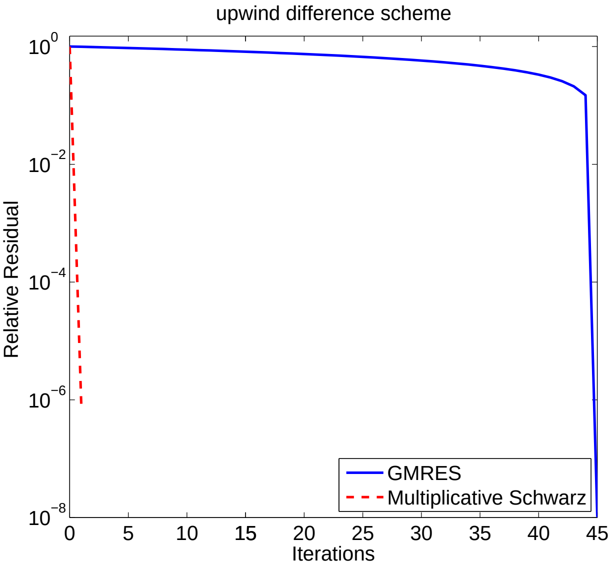
nonsymmetric M-matrix

$$\mathcal{A} = Y \Lambda Y^{-1}$$

$\varepsilon = 10^{-8}$	original	scaled
$\kappa(\mathcal{A})$	10^{10}	10^{3}
$\kappa(Y)$	10^{17}	10^{19}

GMRES versus Schwarz



Multiplicative Schwarz method

results [Echeverría, Liesen, Tichý, Szyld, 2018]

We have shown for $\| \cdot \| = \| \cdot \|_\infty$ that

$$\|x - x^{(k+1)}\| \ \leq \ \rho^k \, \|T\| \, \|x - x^{(0)}\|$$

where

$$\rho \ < \ \frac{\varepsilon}{\varepsilon + \frac{\alpha}{N}}$$

and

$$\|T\| \ \leq \ \rho \quad \text{or} \quad \|T\| \ \leq \ 1\,,$$

depending on the order of domains in the definition of T.

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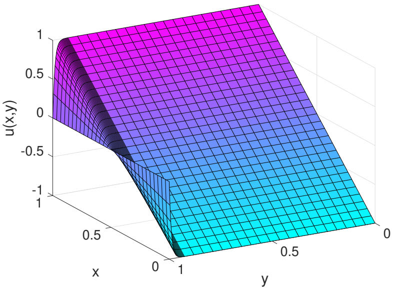
- Rank-one structure of T
- Schwarz as a preconditioner
- Preconditioned GMRES $\rightarrow$ at most 2 iterations

Two-dimensional case

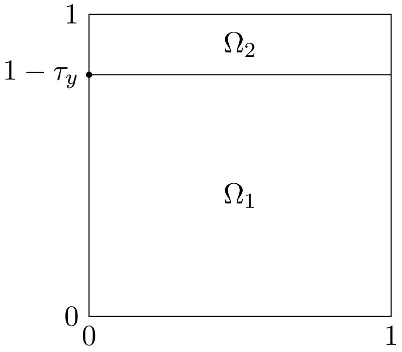
A problems with **one** boundary layer

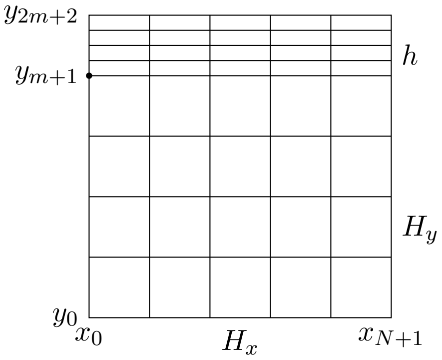
$$-\varepsilon\,\Delta u + u_y = 0$$



$$u(x,y) = (2x-1)\left(\frac{1-e^{(y-1)/\varepsilon}}{1-e^{-1/\varepsilon}}\right)$$

Shishkin mesh





Use the standard upwind difference scheme.

$$-\varepsilon \Delta u + \alpha u_y + \beta u = f$$

$$\mathcal{A} = \left[\begin{array}{cccc|c|cccc}
A_H & B_H & & & & & & & \\
C_H & \ddots & \ddots & & & & & & \\
& \ddots & \ddots & B_H & & & & & \\
& & C_H & A_H & B_H & & & & \\
\hline
& & & C & A & B & & & \\
\hline
& & & & C_h & A_h & B_h & & \\
& & & & & C_h & \ddots & \ddots & \\
& & & & & & \ddots & \ddots & B_h \\
& & & & & & & C_h & A_h
\end{array}\right]$$

$C_H,\ C,\ C_h,\ B_H,\ B,\ B_h,$ .... scalar multiples of I

$A_H,\ A,\ A_h$ .... tridiagonal and Toeplizt

$\mathcal{A}$ .... row and column block diagonally dominant

Application to the convection-diffusion equation

discretized on the Shishkin mesh

In [Echeverría, Liesen, Tichý, 2020] we have shown for $\|\cdot\| = \|\cdot\|_\infty$ that

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where

$$\rho \;<\; \frac{\varepsilon}{\varepsilon + \frac{\alpha}{m}}$$

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depending on the order of domains in the definition of T.

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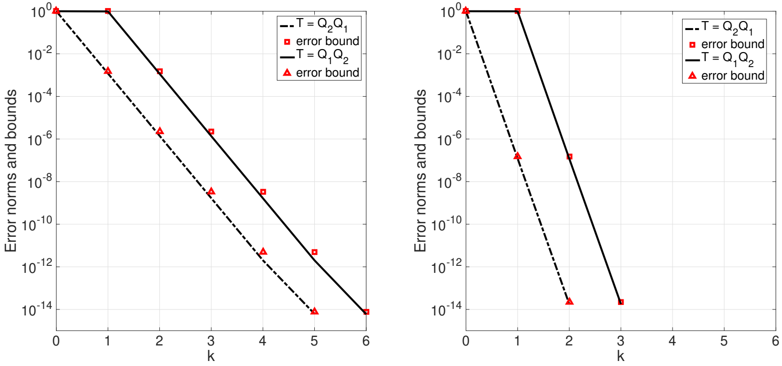
depending on the order of domains in the definition of T.

- Low-rank structure of T
- Schwarz as a preconditioner
- Preconditioned GMRES $\rightarrow$ at most $N+1$ iterations

Tightness of the bound

$$N = 30, \quad m = 20, \quad \mathcal{A} \in \mathbb{R}^{1230 \times 1230}, \quad \alpha = 1, \quad \beta = 0$$



Convergence of multiplicative Schwarz and error bounds
for $\varepsilon = 10^{-4}$ (left) and $\varepsilon = 10^{-8}$ (right)

Conclusions

- Generalization of results [Echeverría, Liesen, Tichý, Szyld, 2018].

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- We analyzed **convergence** of the multiplicative Schwarz method applied to systems with a special block structure

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- Detailed results for **block tridiagonal** matrices.

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$$

- Detailed results for **block tridiagonal** matrices.
- For a particular problem $\rightarrow$ tight and **simple bounds**.

Related papers

- **C. Echeverría, J. Liesen, and P. Tichý**, Analysis of the multiplicative Schwarz method for matrices with a special block structure, Electron. Trans. Numer. Anal. 54, 2021.
- **C. Echeverría, J. Liesen, and R. Nabben**, Block diagonal dominance of matrices revisited: bounds for the norms of inverses and eigenvalue inclusion sets, Linear Algebra Appl. 553, 2018.
- **C. Echeverría, J. Liesen, P. Tichý, and D. Szyld**, Convergence of the multiplicative Schwarz method for singularly perturbed convection-diffusion problems discretized on . . . , Electron. Trans. Numer. Anal. 48, 2018.
- **H-G. Roos, M. Stynes, L. Tobiska**, Robust numerical methods for singularly perturbed differential equations, Springer-Verlag, Berlin, 2008.
- **M. Stynes**, Steady-state convection-diffusion problems, Acta Numer. 14, 2005.

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Thank you for your attention!

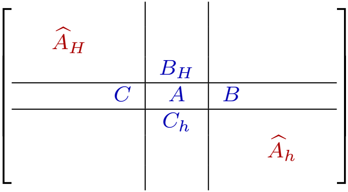
Open problems

Practical implementation issues

To use the iterative scheme

$$x^{(k)} = T\,x^{(k-1)} + v$$

we need to solve **linear systems with submatrices** of



- Schur complement and fast Toeplitz solvers?
- Problems with non-constant coefficients?
- Inexact solvers?

$$x^{(k+1)} = Tx^{(k)} + v, \quad T \equiv I - (P_1 + P_2)$$

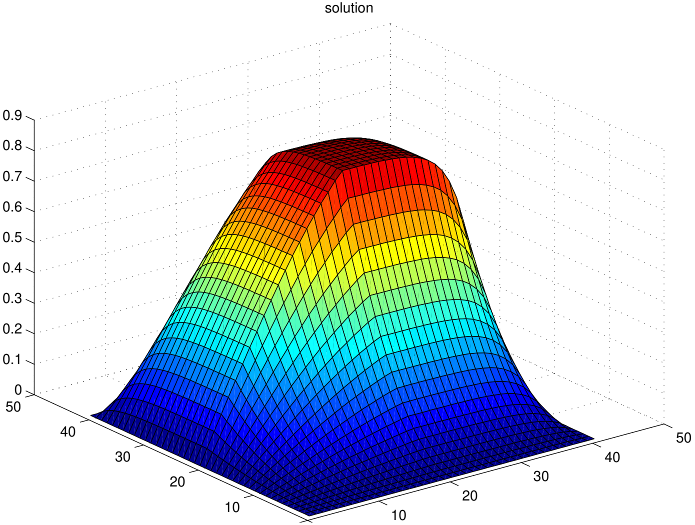
where

$$T = -\begin{bmatrix} 0_{N(m-1)} & & & P^{(1)}_{1:m-1} & \\ & & & P^{(1)}_{m} & \\ & \Pi^{(2)} & I_N & \Pi^{(1)} & \\ & P^{(2)}_{1} & & & \\ & P^{(2)}_{2:m} & & & 0_{N(m-1)} \end{bmatrix}.$$

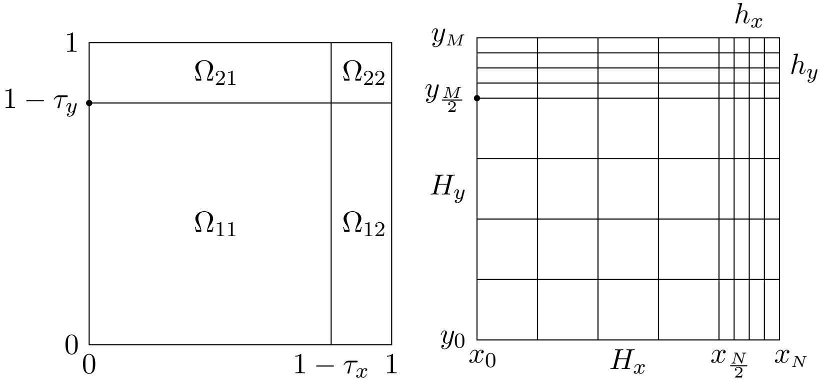
- $\rho(T) \geq 1$
- $I - T$ is nonsingular, T is low rank
- can be used as a preconditioner

Two boundary layers

$$-\varepsilon\,\Delta u \;+\; \alpha_1\, u_x \;+\; \alpha_2\, u_y \;+\; \beta\, u = f$$



Shishkin mesh



- Definition of the multiplicative Schwarz method?
- Structure of $\mathcal{A}$?
- Is T low-rank?