Theory of discontinuous Galerkin method in time dependent domains and applications

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Goal:

Goal: To work out an accurate, efficient and robust, theoretically based numerical method for the solution of nonlinear parabolic problems and compressible Navier-Stokes equations in time dependent domains

Most of the results on the theory and numerical analysis are obtained under the assumption that a space domain Ω is independent of time t.

However, problems in time-dependent domains Ω_t are important in a number of areas of science and technology.

Analysis of stability for a model initial boundary value problem in a time-dependent bounded domain $\Omega_t \subset \boldsymbol{R}^d$, where $t \in [0, T]$, $T > 0$: Find a function $u = u(x,t)$ with $x \in \Omega_t$, $t \in (0, T)$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \operatorname{div}(\beta(u)\nabla u) = g \quad \text{in } \Omega_t,\ t \in (0, T), \tag{1}$$

$$u = u_D \quad \text{on } \partial\Omega_t,\ t \in (0, T), \tag{2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{3}$$

We assume that $f_s \in C^1(\boldsymbol{R})$, $f_s(0) = 0$,

$$|f_s'| \le L_f, \quad s = 1, \dots, d, \tag{4}$$

$$\beta : \mathbb{R} \to [\beta_0, \beta_1], \quad 0 < \beta_0 < \beta_1 < \infty, \tag{5}$$

$$|\beta(u_1) - \beta(u_2)| \le L_\beta |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}. \tag{6}$$

ALE method

$$\mathcal{A}_t : \overline{\Omega}_{\rm ref} \to \overline{\Omega}_t, \quad X \in \overline{\Omega}_{\rm ref} \to x = \mathcal{A}_t(X) \in \overline{\Omega}_t, \quad t \in [0, T]. \tag{7}$$

domain velocity

$$\tilde{\boldsymbol{z}}(X,t) = \frac{\partial}{\partial t}\mathcal{A}_t(X), \ \boldsymbol{z}(x,t) = \tilde{\boldsymbol{z}}(\mathcal{A}_t^{-1}(x), t), \quad t \in [0, T], \ X \in \Omega_{\rm ref}, \ x \in \Omega_t, \tag{8}$$

ALE derivative $D_t f = Df/Dt$ of a function $f = f(x,t)$ for $x \in \Omega_t$ and $t \in [0, T]$ as

$$D_t f(x,t) = \frac{D}{Dt} f(x,t) = \frac{\partial \tilde{f}}{\partial t}(X,t), \tag{9}$$

where $\tilde{f}(X,t) = f(\mathcal{A}_t(X), t)$, $X \in \Omega_{\rm ref}$, and $x = \mathcal{A}_t(X) \in \Omega_t$. The use of the chain rule yields the relation

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{z} \cdot \nabla f, \tag{10}$$

which allows us to reformulate problem (1)–(3) in the ALE form:

Find $u = u(x,t)$ with $x \in \Omega_t,\ t \in (0,T)$ such that

$$\frac{Du}{Dt} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \boldsymbol{z} \cdot \nabla u - \operatorname{div}(\beta(u)\nabla u) = g \quad \text{in } \Omega_t, \qquad t \in (0,T), \tag{11}$$

$$u = u_D \quad \text{on } \partial\Omega_t,\ t \in (0,T), \tag{12}$$

$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{13}$$

ALE-space time DG discretization

partition $0 = t_0 < t_1 < \cdots < t_M = T$ and set $\tau_m = t_m - t_{m-1}$, $I_m = (t_{m-1}, t_m)$, $\bar{I}_m = [t_{m-1}, t_m]$ for $m = 1, \ldots, M$, $\tau = \max_{m=1,\ldots,M} \tau_m$.

The space-time discontinuous Galerkin method (STDGM) allows us to consider an ALE mapping separately on each time interval $[t_{m-1}, t_m)$ for $m = 1, \ldots, M$ and the resulting ALE mapping in $[0, T]$ may be discontinuous at time instants t_m, $m = 1, \ldots, M-1$.

Discrete function spaces

for every $m = 1, \dots, M$ we consider the space

$$S_h^{p,m-1} = \left\{ \varphi \in L^2(\Omega_{t_{m-1}});\ \varphi|_{\hat{K}} \in P^p(\hat{K})\ \forall\, \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}} \right\}, \quad (14)$$

where $p \geq 1$ is an integer and $P^p(\hat{K})$ is the space of all polynomials on $\hat{K}$ of degree $\leq p$.

Further, let $p,\ q \geq 1$ be integers. We set

$$\begin{aligned} & S_{h,\tau}^{p,q} && (15) \\ & = \left\{ \varphi;\ \varphi\left(\mathcal{A}_{h,t}^{m-1}(X), t\right) = \sum_{i=0}^{q} \vartheta_i(X)\, t^i, \quad \vartheta_i \in S_h^{p,m-1}, \right. \\ & \left. X \in \Omega_{t_{m-1}},\ t \in \bar{I}_m,\ m = 1, \dots, M \right\}. \end{aligned}$$

Discrete forms

Diffusion form

$$a_h(u,\varphi,t) := \sum_{K\in\mathcal{T}_{h,t}} \int_K \beta(u)\nabla u\cdot\nabla\varphi\,\mathrm{d}x \tag{16}$$

$$- \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \left(\langle\beta(u)\nabla u\rangle\cdot\mathbf{n}_\Gamma\,[\varphi] + \theta\,\langle\beta(u)\nabla\varphi\rangle\cdot\mathbf{n}_\Gamma\,[u]\right)\mathrm{d}S$$

$$- \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma \left(\beta(u)\nabla u\cdot\mathbf{n}_\Gamma\,\varphi + \theta\beta(u)\nabla\varphi\cdot\mathbf{n}_\Gamma\,u - \theta\beta(u)\nabla\varphi\cdot\mathbf{n}_\Gamma\,u_D\right)\mathrm{d}S,$$

$\theta = -1,\ 0,\ 1$

Interior and boundary penalty

$$J_h(u,\varphi,t) := c_W \sum_{\Gamma\in\mathcal{F}^I_{h,t}} h(\Gamma)^{-1}\int_\Gamma [u]\,[\varphi]\,\mathrm{d}S + c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1}\int_\Gamma u\,\varphi\,\mathrm{d}S, \tag{17}$$

$$A_h(u,\varphi,t) = a_h(u,\varphi,t) + \beta_0\, J_h(u,\varphi,t), \tag{18}$$

Convection forms

$$b_h(u,\varphi,t) := -\sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} f_s(u)\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \tag{19}$$

$$+ \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \mathbf{n}_\Gamma)\,[\varphi]\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(L)}, \mathbf{n}_\Gamma)\,\varphi\,\mathrm{d}S,$$

H – suitable numerical flux to f_s

$$d_h(u,\varphi,t) := -\sum_{K\in\mathcal{T}_{h,t}} \int_K (\boldsymbol{z}\cdot\nabla u)\varphi\,\mathrm{d}x, \tag{20}$$

Right-hand side form

$$\ell_h(\varphi,t) := \sum_{K\in\mathcal{T}_{h,t}} \int_K g\varphi\,\mathrm{d}x + \beta_0\, c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma u_D\,\varphi\,\mathrm{d}S. \tag{21}$$

For a function φ defined in $\bigcup_{m=1}^{M} I_m$ we denote

$$\varphi_m^{\pm} = \varphi(t_m\pm) = \lim_{t \to t_m\pm} \varphi(t), \quad \{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-), \quad (22)$$

if the one-sided limits $\varphi_m^{\pm}$ exist.

ALE-STDG approximate solution

Definition

A function U is an approximate solution of problem (11)–(13), if $U \in S_{h,\tau}^{p,q}$ and

$$\int_{I_m} \left((D_t U, \varphi)_{\Omega_t} + A_h(U, \varphi, t) + b_h(U, \varphi, t) + d_h(U, \varphi, t) \right) \mathrm{d}t$$

$$+(\{U\}_{m-1}, \varphi_{m-1}^{+})_{\Omega_{t_{m-1}}} = \int_{I_m} \ell_h(\varphi, t)\, \mathrm{d}t \quad \forall \varphi \in S_{h,\tau}^{p,q}, \quad (23)$$

$$m = 1, \ldots, M,$$

$$U_0^- \in S_h^{p,0}, \quad (U_0^- - u^0, v_h) = 0 \quad \forall v_h \in S_h^{p,0}. \quad (24)$$

Main stability result

Theorem

There exists a constant $C_1^* > 0$ *independent of* h *and* τ *such that*

$$\|U_m^-\|^2_{\Omega_{t_m}} + \sum_{j=1}^{m} \|\{U\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|U\|^2_{DG,t}\, \mathrm{d}t \quad (25)$$

$$\leq C_1^* \left(\|U_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^{m} \int_{I_j} (\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t})\, \mathrm{d}t \right),$$

$m = 1, \ldots, M,\ h \in (0, \overline{h}).$

Analysis of the stability

Important relations:

a) Boundedness of the Jacobian matrices and Jacobian determinants of the ALE mappings:

$$\left\| \frac{d\mathcal{A}_{h,t}^{m-1}(X)}{dX} \right\| \le C_A^+, \quad \left\| \frac{d(\mathcal{A}_{h,t}^{m-1})^{-1}(x)}{dx} \right\| \le C_A^-, \qquad (26)$$

$$C_J^- \le J(X,t) \le C_J^+, \quad (C_J^+)^{-1} \le J^{-1}(x,t) \le (C_J^-)^{-1}, \quad (27)$$
$$X \in \overline{\Omega}_{t_{m-1}}, \quad x \in \overline{\Omega}_t, \; t \in \overline{I}_m, \; m = 1, \dots, M, \; h \in (0, \overline{h}).$$

b) Multiplicative trace inequality and the inverse inequality:

$$\|v\|^2_{L^2(\partial K)} \le c_M \left(\|v\|_{L^2(K)} |v|_{H^1(K)} + h_K^{-1} \|v\|^2_{L^2(K)} \right), \qquad (28)$$

$v \in H^1(K)$, $K \in \mathcal{T}_{h,t}$, $h \in (0, \overline{h})$, $t \in [0, T]$,

$|v|_{H^1(K)} \le c_I \, h_K^{-1} \|v\|_{L^2(K)}, \quad v \in P^p(K), K \in \mathcal{T}_{h,t}, \; h \in (0, \overline{h}), \; t \in [0, T]$

c) Norms: $\| \cdot \|_{\Omega_t}$ - $L^2(\Omega_t)$ norm

$$\|\varphi\|_{DG,t} = \left(\sum |\varphi|^2_{H^1(K)} + J_h(\varphi, \varphi, t) \right)^{1/2}. \qquad (29)$$

Theorem

There exists a constant $C_{T2} > 0$ *such that*

$$\|U_m^-\|^2_{\Omega_{t_m}} - \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_m} \|U\|^2_{DG,t}\mathrm{d}t \leq C_{T2}\Big(\int_{I_m} \|g\|^2_{\Omega_t}\mathrm{d}t + \int_{I_m} \|u_D\|^2_{DGB,t}\mathrm{d}t + \int_{I_m} \|U\|^2_{\Omega_t}\mathrm{d}t\Big). \tag{30}$$

d) Proof of the boundedness of the problematic term $\int_{I_m} \|U\|^2_{\Omega_t}\mathrm{d}t$ in dependence on the data.

e) Important estimate:

Theorem

There exists a constant $C_{T4} > 0$ independent of h and τ such that

$$\int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t \tag{31}$$
$$\leq C_{T4}\tau_m \left(\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right)\mathrm{d}t \right).$$

f) Now, if (31) is substituted into (30) and the discrete Gronwall inequality is applied, we obtain the unconditional stability of the ALE-STDGM:

There exists a constant $C_1^* > 0$ independent of h, τ, m such that

$$\|U_m^-\|^2_{\Omega_{t_m}} + \sum_{j=1}^{m} \|\{U\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|U\|^2_{DG,t}\,\mathrm{d}t \qquad (32)$$
$$\leq C_1^* \left(\|U_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^{m} \int_{I_j} (\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t})\mathrm{d}t \right),$$
$$m = 1, \ldots, M,\ h \in (0, \overline{h}),$$

Further work: Analysis of error estimates - much more complicated and difficult (M.F. and M. Balázsová)

Application to the simulation of compressible flow interacted with an elastic body

$$\frac{D^A \boldsymbol{w}}{Dt} + \sum_{s=1}^{2} \frac{\partial \boldsymbol{g}_s(\boldsymbol{w})}{\partial x_s} + \boldsymbol{w}\,\mathrm{div}\boldsymbol{z} = \sum_{s=1}^{2} \frac{\partial \boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w})}{\partial x_s}, \qquad (33)$$

Navier-Stokes compressible system, where

$$\boldsymbol{w} = (w_1, \dots, w_4)^T = (\rho, \rho v_1, \rho v_2, E)^T \in \boldsymbol{R}^4,$$

$$\boldsymbol{g}_s(\boldsymbol{w}) = \boldsymbol{f}_s(\boldsymbol{w}) - z_s \boldsymbol{w},$$

$$\boldsymbol{f}_s(\boldsymbol{w}) = (\rho v_s, \rho v_1 v_s + \delta_{1s}\, p, \rho v_2 v_s + \delta_{2s}\, p, (E + p) v_s)^T,$$

$$\boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w}) = \left(0, \tau_{s1}^V, \tau_{s2}^V, \tau_{s1}^V\, v_1 + \tau_{s2}^V\, v_2 + k \partial\theta/\partial x_s\right)^T,$$

$$\boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w}) = \sum_{k=1}^{2} \boldsymbol{K}_{sk}(\boldsymbol{w}) \frac{\partial \boldsymbol{w}}{\partial x_k}, \quad \boldsymbol{f}_s(\boldsymbol{w}) = \frac{D\boldsymbol{f}_s(\boldsymbol{w})}{D\boldsymbol{w}} \boldsymbol{w},$$

$$\tau_{ij}^V = \lambda\, \mathrm{div}\boldsymbol{v}\, \delta_{ij} + 2\mu\, d_{ij}(\boldsymbol{v}),\ \ d_{ij}(\boldsymbol{v}) = \left(\partial v_i/\partial x_j + \partial v_j/\partial x_i\right)/2$$

Thermodynamical relations

$$p = (\gamma - 1)(E - \rho|\boldsymbol{v}|^2/2), \quad \theta = \left(E/\rho - |\boldsymbol{v}|^2/2\right)/c_v.$$

Initial and boundary conditions

Notation: ρ - density,

p - pressure,

E - total energy,

$\boldsymbol{v} = (v_1, v_2)$ - velocity,

θ - absolute temperature,

$\gamma > 1$ - Poisson adiabatic constant,

$c_v > 0$ - specific heat at constant volume,

$\mu > 0, \lambda = -2\mu/3$ - viscosity coefficients,

$k > 0$ - heat conduction

Dynamic elasticity problem

Consider an elastic body represented by a bounded domain $\Omega^b \subset \mathbb{R}^2$ with boundary $\partial\Omega^b = \Gamma_D^b \cup \Gamma_N^b$, we seek for a displacement function $\boldsymbol{y} : Q_T = \Omega^b \times [0, T] \to \mathbb{R}^2$ such that

$$\rho^b \frac{\partial^2 \boldsymbol{y}}{\partial t^2} + c_M \rho^b \frac{\partial \boldsymbol{y}}{\partial t} - \text{div} \boldsymbol{P}(\nabla \boldsymbol{y}) = \boldsymbol{f} \quad \text{in } \Omega^b \times [0, T], \tag{34}$$

$$\boldsymbol{y} = \boldsymbol{y}_D \quad \text{in } \Gamma_D^b \times [0, T], \tag{35}$$

$$\boldsymbol{P}(\nabla \boldsymbol{y}) \cdot \boldsymbol{n} = \boldsymbol{g}_N \quad \text{in } \Gamma_N^b \times [0, T], \tag{36}$$

$$\boldsymbol{y}(\cdot, 0) = \boldsymbol{y}_0, \quad \frac{\partial \boldsymbol{y}}{\partial t}(\cdot, 0) = \boldsymbol{z}_0 \quad \text{in } \Omega^b, \tag{37}$$

$\boldsymbol{f}$ - outer volume force, $\rho^b > 0$ - material density

$\boldsymbol{P}$ - (Piola-Kirchhoff) stress tensor

Blue term - structural damping

Linear elasticity

Nonlinear elasticity:

St. Venant-Kirchhoff material

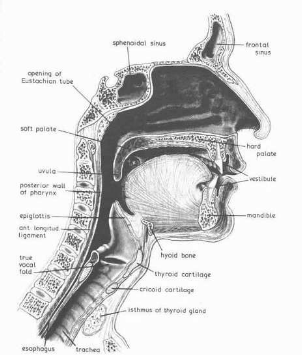
The Piola-Kirchhoff stress tensor and the second Piola-Kirchhoff stress tensor:

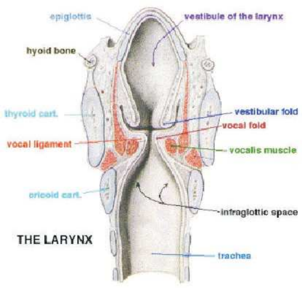
$$\boldsymbol{P}(\boldsymbol{F}) = \boldsymbol{F}\boldsymbol{\Sigma}, \quad \boldsymbol{\Sigma} = \lambda^b \mathrm{tr}(\boldsymbol{E})\boldsymbol{I} + 2\mu^b \boldsymbol{E} \tag{38}$$

Neo-Hookean material

$\boldsymbol{P}(\boldsymbol{F}) = \mu^b(\boldsymbol{F} - \boldsymbol{F}^{-T}) + \lambda^b \log(\det \boldsymbol{F})\boldsymbol{F}^{-T}$

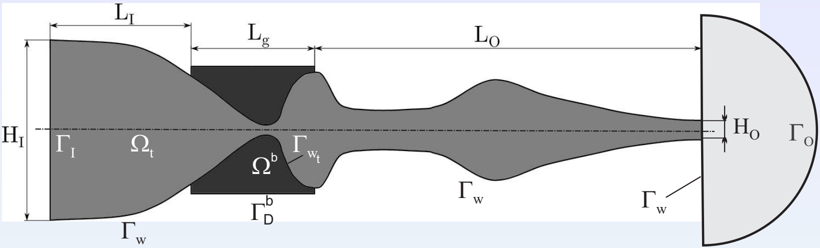
Application: The FSI is applied to the numerical simulation of flow-induced vibrations of vocal folds



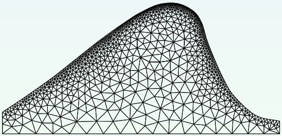


Scheme of the vocal tract in the sagital cut and a detail in the coronal cut

The above pictures inspire us to the channel representing the human glottis coupled with elastic domains showing vocal folds.



Geometry of the computational domain at time $t = 0$ and the description of its size: $L_I = 20.0$ mm, $L_g = 17.5$ mm, $L_O = 55.0$ mm, $H_I = 25.5$ mm, $H_O = 2.76$ mm.



Model of vocal folds - computational mesh.

Interaction of compressible flow and Neo-Hookean elasticity model of vocal folds

Vocal folds vibrations6

Coanda effect - main streams are attached to walles

Further goals:

- analysis of error estimate of the STDGM in time-dependent domains
- further complicated numerical experiments
- applications to sophisticated practical FSI problems
- analysis of the acoustic signal

THANK YOU FOR YOUR ATTENTION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