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Some aspects of variational time discretisations of higher order and higher regularity

10th Workshop Dresden-Prague on Numerical Analysis
Děčín, November 4-5, 2022

Contents

Initial-boundary value problem

spatial domain $\Omega \subset \mathbb{R}^d$
time intervall $I = (0, T)$ with $T > 0$

time-dependent convection-diffusion-reaction equation (with time-constant coefficients)

$$\partial_t u(t,x) - \mu(x)\triangle u(t,x) + b(x) \cdot \nabla u(t,x) + c(x)u(t,x) = f(t,x), \qquad t \in (0,T),\ x \in \Omega,$$

homogeneous Dirichlet boundary conditions

$$u(t,x) = 0, \qquad t \in (0,T),\ x \in \partial\Omega,$$

initial condition

$$u(0,x) = u_0, \qquad x \in \Omega$$

Parabolic problem

function spaces

- $V = H_0^1(\Omega)$ with dual space $V' = H^{-1}(\Omega)$
- $H = L^2(\Omega)$

Weak formulation

Find $u \in \{v \in L^2(I,V) : \partial_t v \in L^2(I,V')\}$ with $u(0) = u_0$ such that

$$\langle \partial_t u(t), v\rangle + a\big(u(t), v\big) = \langle f(t), v\rangle \qquad \text{for a.e. } t \in I = (0,T],\ \forall v \in V,$$

where $f \in L^2(I,V')$ and $u_0 \in H$.

notation and assumptions

- coercive and continuous bilinear form a defined on $H_0^1(\Omega)$, independent of time t
- $\langle\cdot,\cdot\rangle : H^{-1}(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ duality pairing, extension of inner product $(\cdot,\cdot)$ in $L^2(\Omega)$
- $\|\cdot\|$ norm in $L^2(\Omega)$

Semi-discretization in space

V_h finite dimensional subspace of V;   assume $f \in C(\overline{I}, V')$

Semi-discretization in space

Find $u_h \in C^1(\overline{I}, V_h)$ with $u_h(0) = u_{h,0}$ such that

$$\big(\partial_t u_h(t), v_h\big) + a\big(u_h(t), v_h\big) = \big\langle f(t), v_h \big\rangle \qquad \forall t \in \overline{I},\ \forall v_h \in V_h,$$

where $u_{h,0} \in V_h$ is an approximation of the initial value u_0.

writing $u_h(\cdot) = \sum_{i=1}^{m} U_{h,i}(\cdot)\varphi_i$ w.r.t. basis $\{\varphi_1, \ldots, \varphi_m\}$ of V_h results in ode system

$$MU_h'(t) + AU_h(t) = F(t) \quad \forall t \in I, \qquad U_h(0) = U_{h,0}$$

here:
- vector field on rhs $F_j(\cdot) = \big\langle f(\cdot), \varphi_j \big\rangle, \qquad j = 1, \ldots, m$
- mass matrix M $M_{ij} = \big(\varphi_j, \varphi_i\big), \qquad i, j = 1, \ldots, m$
- stiffness matrix A $A_{ij} = a\big(\varphi_j, \varphi_i\big), \qquad i, j = 1, \ldots, m$

Variational Time Discretization (VTD) methods – Basic idea

decomposition of time interval $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$

- subintervals $I_n = (t_{n-1}, t_n]$, $1 \le n \le N$
- their lengths $\tau_n = t_n - t_{n-1}$, $1 \le n \le N$, $\tau = \max\limits_{1\le n\le N} \tau_n$
- jump of v at t_n $[v]_n = v(t_n^+) - v(t_n^-)$

Short description of VTD_k^r framework (locally on I_n) with parameters $r, k \in \mathbb{Z}$, $0 \le k \le r$

(local) ansatz space: P_r,

(local) test space: P_{r-k},

if $k \ge 1$: initial condition,

if $k \ge 2$: $\mathrm{ODE}^{(i)}$ in t_n^-, $i = 0, \dots, \left\lfloor \frac{k}{2} \right\rfloor - 1$,

if $k \ge 3$: $\mathrm{ODE}^{(i)}$ in t_{n-1}^+, $i = 0, \dots, \left\lfloor \frac{k-1}{2} \right\rfloor - 1$.

Here, $\mathrm{ODE}^{(i)}$ means that the discrete solution fulfills the ith derivative of the ode system.

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$$U_{\tau h}|_{I_n} \in P_r(I_n, \mathbb{R}^m) : \quad \int_{I_n} \left(MU'_{\tau h} + AU_{\tau h}\right) \cdot V_{\tau h} \,\mathrm{d}t + \ldots = \int_{I_n} F \cdot V_{\tau h} \,\mathrm{d}t \quad \forall V_{\tau h} \in P_{r-k}(I_n, \mathbb{R}^m)$$

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Here, $\mathrm{ODE}^{(i)}$ means that the discrete solution fulfills the ith derivative of the ode system.

$U_{\tau h}|_{I_n} \in P_r\big(I_n, \mathbb{R}^m\big)$: $U_{\tau h}(t_{n-1}^+) = U_{\tau h}(t_{n-1}^-)$

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decomposition of time interval $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$

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Here, $\mathrm{ODE}^{(i)}$ means that the discrete solution fulfills the ith derivative of the ode system.

$U_{\tau h}|_{I_n} \in P_r\big(I_n, \mathbb{R}^m\big)$: $\qquad MU_{\tau h}^{(i+1)}(\tilde{t}) + AU_{\tau h}^{(i)}(\tilde{t}) = F^{(i)}(\tilde{t}), \quad \tilde{t} \in \{t_n^-, t_{n-1}^+\}$

Variational Time Discretization (VTD) methods – $\mathscr{I}_n$-$\mathrm{VTD}_k^r(g)$

integrator $\mathscr{I}_n$ approximates integral over I_n

$\mathscr{I}_n$-$\mathbf{VTD}_k^r(g)$ method (local problem on I_n)

Find $u_{\tau h}|_{I_n} \in P_r(I_n, V_h)$ such that

$$u_{\tau h}(t_{n-1}^+) = u_{\tau h}(t_{n-1}^-), \qquad \text{if } k \geq 1,$$

$$\big(\partial_t^{i+1} u_{\tau h}(t_n^-), v_h\big) + a\big(\partial_t^i u_{\tau h}(t_n^-), v_h\big) = \big\langle g^{(i)}(t_n^-), v_h\big\rangle \quad \forall v_h \in V_h, \quad \text{if } k \geq 2,\ i = 0, \ldots, \left\lfloor \tfrac{k}{2} \right\rfloor - 1,$$

$$\big(\partial_t^{i+1} u_{\tau h}(t_{n-1}^+), v_h\big) + a\big(\partial_t^i u_{\tau h}(t_{n-1}^+), v_h\big) = \big\langle g^{(i)}(t_{n-1}^+), v_h\big\rangle \quad \forall v_h \in V_h, \quad \text{if } k \geq 3,\ i = 0, \ldots, \left\lfloor \tfrac{k-1}{2} \right\rfloor - 1,$$

$$\mathscr{I}_n\Big[\big(\partial_t u_{\tau h}, v_{\tau h}\big) + a\big(u_{\tau h}, v_{\tau h}\big)\Big] + \delta_{0k}\big([u_{\tau h}]_{n-1}, v_{\tau h}(t_{n-1}^+)\big) = \mathscr{I}_n\big[\langle g, v_{\tau h}\rangle\big] \quad \forall v_{\tau h} \in P_{r-k}(I_n, V_h)$$

where $u_{\tau h}(0^-) \in V_h$ is an approximation of $u(0) = u_0$ and g is some approximation of f.

Variational Time Discretization (VTD) methods – $\mathscr{I}_n$-$\mathrm{VTD}_k^r(g)$

integrator $\mathscr{I}_n$ approximates integral over I_n

$\mathscr{I}_n$-$\mathrm{VTD}_k^r(g)$ method (local problem on I_n) $k=0 \mathrel{\widehat{=}} \mathrm{dG}(r)$ method

Find $u_{\tau h}|_{I_n} \in P_r(I_n, V_h)$ such that

$$u_{\tau h}(t_{n-1}^+) = u_{\tau h}(t_{n-1}^-), \qquad \text{if } k \geq 1,$$

$$\big(\partial_t^{i+1} u_{\tau h}(t_n^-), v_h\big) + a\big(\partial_t^i u_{\tau h}(t_n^-), v_h\big) = \big\langle g^{(i)}(t_n^-), v_h\big\rangle \quad \forall v_h \in V_h,$$
$$\text{if } k \geq 2,\ i=0,\dots,\left\lfloor \tfrac{k}{2} \right\rfloor - 1,$$

$$\big(\partial_t^{i+1} u_{\tau h}(t_{n-1}^+), v_h\big) + a\big(\partial_t^i u_{\tau h}(t_{n-1}^+), v_h\big) = \big\langle g^{(i)}(t_{n-1}^+), v_h\big\rangle \quad \forall v_h \in V_h,$$
$$\text{if } k \geq 3,\ i=0,\dots,\left\lfloor \tfrac{k-1}{2} \right\rfloor - 1,$$

$$\mathscr{I}_n\Big[\big(\partial_t u_{\tau h}, v_{\tau h}\big) + a\big(u_{\tau h}, v_{\tau h}\big)\Big] + \delta_{0k}\big([u_{\tau h}]_{n-1}, v_{\tau h}(t_{n-1}^+)\big) = \mathscr{I}_n\big[\langle g, v_{\tau h}\rangle\big] \quad \forall v_{\tau h} \in P_{r-k}(I_n, V_h)$$

where $u_{\tau h}(0^-) \in V_h$ is an approximation of $u(0) = u_0$ and g is some approximation of f.

Variational Time Discretization (VTD) methods – $\mathscr{I}_n$-$\mathrm{VTD}_k^r(g)$

integrator $\mathscr{I}_n$ approximates integral over I_n

$\mathscr{I}_n$-$\mathrm{VTD}_k^r(g)$ method (local problem on I_n) $k = 1 \mathrel{\widehat{=}} \mathrm{cGP}(r)$ method

Find $u_{\tau h}|_{I_n} \in P_r(I_n, V_h)$ such that

$$u_{\tau h}(t_{n-1}^+) = u_{\tau h}(t_{n-1}^-), \qquad \text{if } k \geq 1,$$

$$\big(\partial_t^{i+1} u_{\tau h}(t_n^-), v_h\big) + a\big(\partial_t^i u_{\tau h}(t_n^-), v_h\big) = \big\langle g^{(i)}(t_n^-), v_h\big\rangle \quad \forall v_h \in V_h,$$
$$\text{if } k \geq 2,\ i = 0, \ldots, \left\lfloor \tfrac{k}{2} \right\rfloor - 1,$$

$$\big(\partial_t^{i+1} u_{\tau h}(t_{n-1}^+), v_h\big) + a\big(\partial_t^i u_{\tau h}(t_{n-1}^+), v_h\big) = \big\langle g^{(i)}(t_{n-1}^+), v_h\big\rangle \quad \forall v_h \in V_h,$$
$$\text{if } k \geq 3,\ i = 0, \ldots, \left\lfloor \tfrac{k-1}{2} \right\rfloor - 1,$$

$$\mathscr{I}_n\Big[\big(\partial_t u_{\tau h}, v_{\tau h}\big) + a\big(u_{\tau h}, v_{\tau h}\big)\Big] + \delta_{0k}\big([u_{\tau h}]_{n-1}, v_{\tau h}(t_{n-1}^+)\big) = \mathscr{I}_n\big[\langle g, v_{\tau h}\rangle\big] \quad \forall v_{\tau h} \in P_{r-1}(I_n, V_h)$$

where $u_{\tau h}(0^-) \in V_h$ is an approximation of $u(0) = u_0$ and g is some approximation of f.

Special quadrature formulas

$\mathcal{I}_k^r$ interpolation operator (locally on I_n)

Let $\mathcal{I}_k^r : C^{\lfloor \frac{k}{2} \rfloor}(\bar{I}_n, X) \to P_r(\bar{I}_n, X)$, for some Banach space X over $\mathbb{R}$, be defined by

$$
\begin{aligned}
(\mathcal{I}_k^r v)^{(i)}(t_{n-1}^+) &= v^{(i)}(t_{n-1}^+), && \text{if } k \geq 1,\ i = 0, \ldots, \left\lfloor \tfrac{k-1}{2} \right\rfloor, \\
(\mathcal{I}_k^r v)^{(i)}(t_n^-) &= v^{(i)}(t_n^-), && i = 0, \ldots, \left\lfloor \tfrac{k}{2} \right\rfloor, \\
\mathcal{I}_k^r v(t_{n,i}) &= v(t_{n,i}), && i = 1, \ldots, r-k,
\end{aligned}
$$

with $t_{n,i} = \frac{t_n + t_{n-1}}{2} + \frac{\tau_n}{2} \hat{t}_i \in I_n$, $i = 1, \ldots, r-k$, where $\hat{t}_i \in (-1, 1)$ are the zeros of the $(r-k)$th Jacobi-polynomial w.r.t. the weight $(1+\hat{t})^{\lfloor \frac{k-1}{2} \rfloor + 1}(1-\hat{t})^{\lfloor \frac{k}{2} \rfloor + 1}$.

associated interpolatory **quadrature formula**

$$
Q_k^r[\varphi] := \int_{I_n} \mathcal{I}_k^r \varphi(t)\,\mathrm{d}t \approx \int_{I_n} \varphi(t)\,\mathrm{d}t
$$

exact for polynomials up to (the best possible) degree $2r-k$

Postprocessing

$\mathscr{I}_n = Q_k^r$ enables simple and beneficial postprocessing

Postprocessing $Q_k^r\text{-VTD}_k^r(g) \rightsquigarrow Q_k^r\text{-VTD}_{k+2}^{r+1}(g)$

Let $u_{\tau h}$ solve $Q_k^r\text{-VTD}_k^r(g)$. For $n = 1, \ldots, N$ set

$$\widetilde{u}_{\tau h}\big|_{I_n} = u_{\tau h}\big|_{I_n} + \widetilde{a}_n \vartheta_n, \qquad \vartheta_n \in P_{r+1}(I_n, \mathbb{R}),$$

where ϑ_n vanishes in the $(r+1)$ quadrature points of Q_k^r and satisfies $\vartheta_n^{(\lfloor \frac{k}{2} \rfloor + 1)}(t_n^-) = 1$ while the function $\widetilde{a}_n \in V_h$ is defined by

$$(\widetilde{a}_n, v_h) = \left\langle g^{(\lfloor \frac{k}{2} \rfloor)}(t_n^-), v_h \right\rangle - a\left(\partial_t^{\lfloor \frac{k}{2} \rfloor} u_{\tau h}(t_n^-), v_h\right) - \left(\partial_t^{\lfloor \frac{k}{2} \rfloor + 1} u_{\tau h}(t_n^-), v_h\right) \qquad \forall v_h \in V_h.$$

Moreover, let $\widetilde{u}_{\tau h}(0^-) = u_{\tau h}(0^-)$. Then, $\widetilde{u}_{\tau h}$ solves $Q_k^r\text{-VTD}_{k+2}^{r+1}(g)$.

note:
- (local) polynomial degree and global temporal differentiability improved by one
- in particular the function value at t_n^- is preserved

Postprocessing – alternative calculation

assume $g \in C^{\lfloor \frac{k-1}{2} \rfloor}(\overline{I}, V') \quad \Rightarrow \quad u_{\tau h} \in C^{\lfloor \frac{k-1}{2} \rfloor}(\overline{I}, V_h)$

Alternative calculation of $\widetilde{a}_n$

$$\widetilde{a}_n = \frac{-1}{\vartheta_n^{(\lfloor \frac{k-1}{2} \rfloor + 1)}(t_{n-1}^+)} \Big(\partial_t^{\lfloor \frac{k-1}{2} \rfloor + 1} u_{\tau h}(t_{n-1}^+) - \partial_t^{\lfloor \frac{k-1}{2} \rfloor + 1} \widetilde{u}_{\tau h}(t_{n-1}^-) \Big), \qquad n > 1$$

postprocessing makes the lowest order derivative continuous that is not contiuous by construction of the method

Cascadic interpolation

problem: quadrature rule and method do not match after one postprocessing step

$$\underbrace{Q_k^r\text{-VTD}_k^r(g)}_{\substack{\text{quadrature rule } Q_k^r \\ \text{matches method } \text{VTD}_k^r}} \qquad \rightsquigarrow \qquad \underbrace{Q_k^r\text{-VTD}_{k+2}^{r+1}(g)}_{\substack{\text{quadrature rule } Q_k^r \\ \text{does } \underline{\text{not}} \text{ match method } \text{VTD}_{k+2}^{r+1}}}$$

idea: choose appropriate g that allows change of quadrature rule

$\boxed{g = \mathcal{I}_{k+2}^{r+1} f}$:

$$Q_k^r\text{-VTD}_k^r(\mathcal{I}_{k+2}^{r+1} f) \rightsquigarrow Q_k^r\text{-VTD}_{k+2}^{r+1}(\mathcal{I}_{k+2}^{r+1} f) \underset{(1)}{\widehat{=}} Q_{k+2}^{r+1}\text{-VTD}_{k+2}^{r+1}(\mathcal{I}_{k+2}^{r+1} f) \underset{(2)}{\widehat{=}} Q_{k+2}^{r+1}\text{-VTD}_{k+2}^{r+1}(f)$$

(1) Q_k^r and Q_{k+2}^{r+1} both are $(2r-k)$-exact, integrands are in P_{2r-k}

(2) $\mathcal{I}_{k+2}^{r+1}$ preserves quadrature points of Q_{k+2}^{r+1}

generalization: cascadic interpolation to enable multiple postprocessing steps

$\boxed{g = \mathcal{C}_k^r f := \mathcal{I}_k^r \circ \mathcal{I}_{k+2}^{r+1} \circ \ldots \circ \mathcal{I}_{2r-k}^{2r-k} f}$:

$$Q_k^r\text{-VTD}_k^r(\mathcal{C}_k^r f) \rightsquigarrow Q_k^r\text{-VTD}_{k+2}^{r+1}(\mathcal{C}_k^r f) \mathrel{\widehat{=}} Q_{k+2}^{r+1}\text{-VTD}_{k+2}^{r+1}(\mathcal{C}_{k+2}^{r+1} f)$$

Discrete initial values

restriction to $\mathscr{I}_n = \int_{I_n}$ and $g \in \{f, \mathcal{I}_k^r f, \mathcal{C}_k^r f\}$; $\ell := \left\lfloor \frac{k}{2} \right\rfloor$

Special choice of discrete initial values

Determine $\partial_t^i u_{\tau h}(t_0^-) \in V_h$, $i = \ell, \dots, 0$, by

$$\partial_t^\ell u_{\tau h}(t_0^-) := \widetilde{P}_h^0 \partial_t^\ell u(0^+),$$

$\partial_t^i u_{\tau h}(0^-) \in V_h$ with $i = \ell - 1, \dots, 0$:

$$a\big(\partial_t^i u_{\tau h}(0^-), v_h\big) = \langle g^{(i)}(0^+), v_h \rangle_{V',V} - \big(\partial_t^{i+1} u_{\tau h}(0^-), v_h\big) \quad \forall v_h \in V_h,$$

where $\widetilde{P}_h^0$ is some projection operator into V_h.

- assuming suitable compatibility conditions: $\partial_t^\ell u(0^+)$ recursively determined by data
- typical choices for $\widetilde{P}_h^0$: H-orthogonal projection, Ritz projection

Final error estimates

V_h: consists of elements that contain polynomials of degree κ
remember: $\|\cdot\|$ is L^2-norm

Global L^2-error in the H-norm

$$\int_0^{t_n} \big\| u - u_{\tau h} \big\|^2 \,\mathrm{d}t + \big\| \big(u - u_{\tau h}\big)(t_n^-) \big\|^2 \le C(f,u)\Big(\tau^{2(r+1)} + h^{2(\kappa+1)}\Big)$$

also optimal estimate for ith derivative, $0 \le i \le \ell$

$$\int_0^{t_n} \big\| \big(u - u_{\tau h}\big)^{(i)} \big\|^2 \,\mathrm{d}t + \big\| \big(u - u_{\tau h}\big)^{(i)}(t_n^-) \big\|^2 \le C(f,u)\Big(\tau^{2(r-i+1)} + h^{2(\kappa+1)}\Big)$$

Details on the implementation and the investigated (semi-)norms

(semi-)norms under consideration:

$$\|v\|_{L^2(L^2)} = \left(\int_I \int_\Omega \left|v(t,x)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2}, \qquad |v|_{H^1(L^2)} = \left(\int_I \int_\Omega \left|\partial_t v(t,x)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2},$$

$$\|v\|_{\ell^\infty(L^2)} = \max_{1\le n\le N} \left(\int_\Omega \left|v(t_n^-,x)\right|^2 \mathrm{d}x\right)^{1/2}$$

implementation:

- programming language **Julia**
- floating point data type `BigFloat` with 512 bits

Test problem

Instationary convection-diffusion-reaction problem

$$
\begin{aligned}
u_t(t,x) - u_{xx}(t,x) + (1+x^2)u_x(t,x) + (1+2x)u(t,x) &= f(t,x) && \text{for } (t,x) \in (0,2)\times(0,1),\\
u(t,0) = u(t,1) &= 0 && \text{for } t \in (0,2),\\
u(0,x) &= x^2(1-x) && \text{for } x \in (0,1)
\end{aligned}
$$

with f chosen such that

$$u(t,x) = x(1-x)\big(x\cos(t) - \sin(2t)\big)$$

is the exact solution

- spatial discretization: continuous, piecewise cubic finite elements, spatial interval $(0,1)$ decomposed into 10 uniform subintervals $\Rightarrow$ spatial error is negligible
- temporal discretization: different versions of Q_k^6-$\mathrm{VTD}_k^6(\cdot)$, $0 \le k \le 6$, uniform time meshes with N subintervals, $N \in \{256, 512\}$

Errors in $L^2(L^2)$ and $H^1(L^2)$

Q_k^6-$\mathrm{VTD}_k^6(\cdot)$, $0 \le k \le 6$, on uniform time mesh with 256 subintervals
eoc calculated from errors on uniform time meshes with $N \in \{256, 512\}$

	$\lVert u - u_{\tau h}\rVert_{L^2(L^2)}$		$\lVert u - \widetilde{u}_{\tau h}\rVert_{L^2(L^2)}$		$\lvert u - u_{\tau h}\rvert_{H^1(L^2)}$		$\lvert u - \widetilde{u}_{\tau h}\rvert_{H^1(L^2)}$	
k	error	eoc	error	eoc	error	eoc	error	eoc
0	9.616e-22	7.000	6.855e-25	7.999	4.548e-18	6.000	1.694e-21	6.998
1	1.007e-21	7.000	9.741e-25	8.000	2.342e-18	6.000	1.781e-21	7.000
2	1.645e-21	7.000	1.801e-24	7.999	3.460e-18	6.000	2.903e-21	6.999
3	2.418e-21	7.000	3.307e-24	8.000	3.662e-18	6.000	4.275e-21	7.000
4	4.836e-21	7.000	8.527e-24	7.999	6.190e-18	6.000	8.537e-21	6.999
5	1.026e-20	7.000	2.201e-23	8.000	9.650e-18	6.000	1.813e-20	7.000
6	4.139e-20	7.000	4.543e-21	6.999	2.228e-17	6.000	4.833e-20	6.999

postprocessing improves order by one
no improvement (since superconvergence order $2r - k + 1$ is already reached)

Errors in $\ell^\infty(L^2)$ for solution and derivative

Q_k^6-$\mathrm{VTD}_k^6(\cdot)$, $0 \le k \le 6$, on uniform time mesh with 256 subintervals
eoc calculated from errors on uniform time meshes with $N \in \{256, 512\}$

	$\|u - u_{\tau h}\|_{\ell^\infty(L^2)}$				$\|\partial_t(u - u_{\tau h})\|_{\ell^\infty(L^2)}$			
	standard		cascade		standard		cascade	
k	error	eoc	error	eoc	error	eoc	error	eoc
0	2.022e-28	9.517	6.809e-40	12.997	4.301e-18	6.000	4.301e-18	6.000
1	6.102e-28	9.404	1.137e-36	12.000	7.987e-18	6.000	7.987e-18	6.000
2	9.839e-28	9.225	1.884e-33	10.997	1.236e-24	8.339	2.195e-32	10.997
3	5.295e-27	9.242	2.663e-30	10.000	3.989e-24	8.257	3.103e-29	10.000
4	1.665e-25	8.983	3.731e-27	8.996	1.136e-23	8.248	4.346e-26	8.996
5	2.828e-23	8.000	4.317e-24	8.000	3.496e-22	7.989	5.029e-23	8.000
6	4.939e-21	6.995	4.939e-21	6.995	5.754e-20	6.995	5.754e-20	6.995

cascadic interpolation provides (optimal) superconvergence order $2r - k + 1$
superconvergence for $k \in \{0, 1\}$ after postprocessing only (since no collocation at t_n^- before)

Discrete initial values – “downward” vs. “upward” generation

Discrete initial values (determined in “downward” direction)

$$\partial_t^{\lfloor \frac{k}{2} \rfloor} u_{\tau h}(t_0^-) := \widetilde{P}_h^0 \partial_t^{\lfloor \frac{k}{2} \rfloor} u(0^+) \in V_h,$$

$\partial_t^i u_{\tau h}(0^-) \in V_h$ with $i = \lfloor \frac{k}{2} \rfloor - 1, \dots, 0$:

$$a\big(\partial_t^i u_{\tau h}(0^-), v_h\big) = \big\langle g^{(i)}(0^+), v_h \big\rangle_{V',V} - \big(\partial_t^{i+1} u_{\tau h}(0^-), v_h\big) \quad \forall v_h \in V_h$$

Discrete initial values (determined in “upward” direction)

$$\partial_t^0 u_{\tau h}(t_0^-) := \widetilde{P}_h^0 u_0 \in V_h,$$

$\partial_t^i u_{\tau h}(0^-) \in V_h$ with $i = 1, \dots, \lfloor \frac{k}{2} \rfloor$:

$$\big(\partial_t^i u_{\tau h}(0^-), v_h\big) = \big\langle g^{(i-1)}(0^+), v_h \big\rangle_{V',V} - a\big(\partial_t^{i-1} u_{\tau h}(0^-), v_h\big) \quad \forall v_h \in V_h$$

Deficit compared to optimal order $r + 1$ in $L^2(L^2)$

heat equation in $(0,3) \times (0,1)$ with solution $u(t,x) = x \cos\left(\frac{3\pi}{2}\right) \sin(3t)$

discrete initial values determined in “upward” direction
continuous, piecewise P_r-finite elements

r	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$	$k=12$	$k=13$
5	0.000								
6	0.000	0.000							
7	0.000	0.000	0.999						
8	0.000	0.000	0.735	0.228					
9	0.000	0.000	1.000	0.500	1.998				
10	0.000	0.000	0.808	0.302	1.138	0.640			
11	0.000	0.000	1.001	0.501	1.999	1.499	2.997		
12	0.000	0.000	0.847	0.329	1.184	0.687	2.014	1.514	
13	0.000	0.000	1.000	0.501	2.000	1.500	2.998	2.498	3.996

eoc calculated from errors on uniform meshes with $\tau = 3h = 3/N$ and $N \in \{128, 256\}$