Stopping criteria for coarsest grid solver in multigrid V-cycle method

Petr Vacek, Erin Carson, Kirk M. Soodhalter
Charles University in Prague

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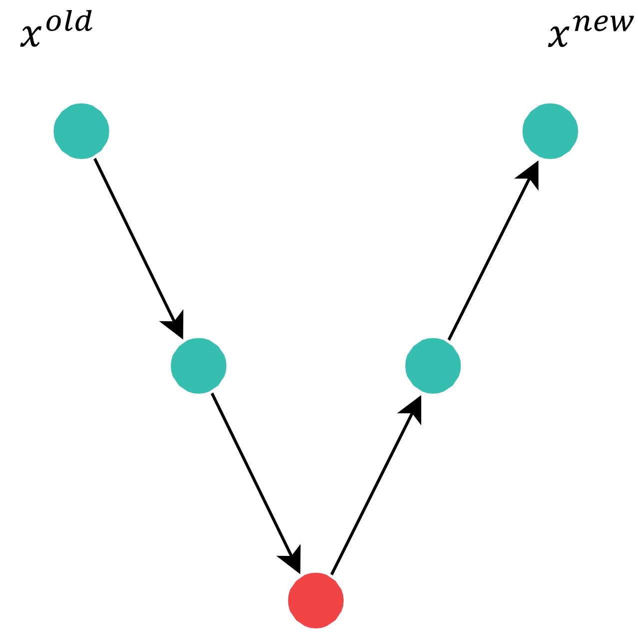
FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Introduction

Find x : $Ax = b$.



Find y_0: $A_0 y_0 = f_0$.

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- <u>smoothing</u>
  - few iterations of stationary iterative method (Jacobi, Gauss-Seidel)

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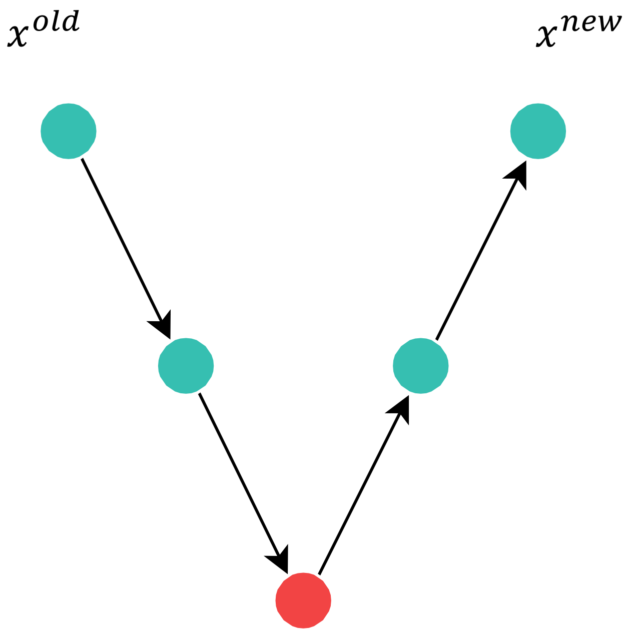

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- few iterations of stationary iterative method (Jacobi, Gauss-Seidel)

solving
- direct solver based on LU decomposition
- iterative solver (Krylov subspace method) [Huber, 2019]
- direct solver based on low rank approximation [Buttari et al., 2021]

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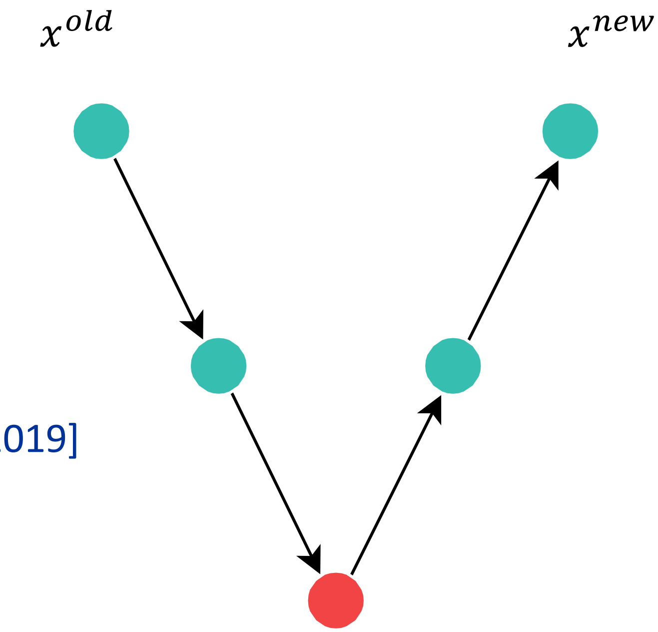

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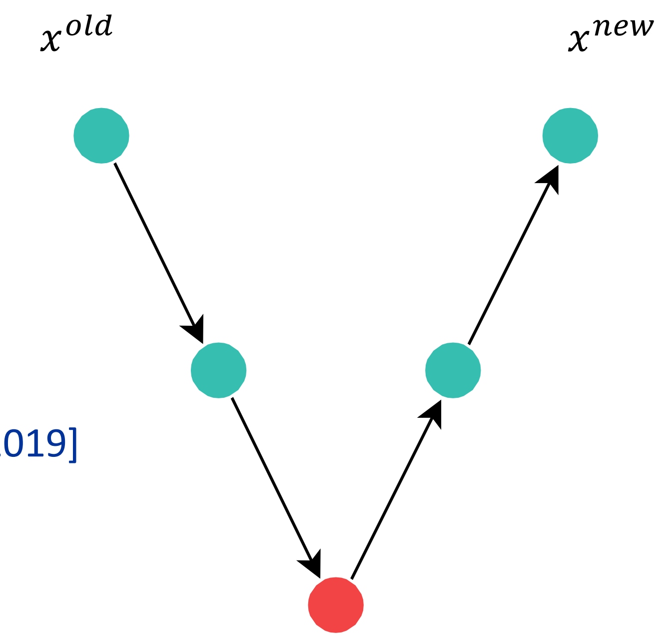

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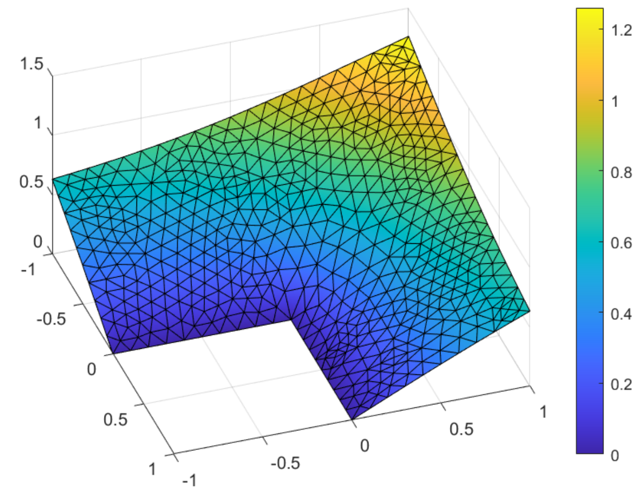
<u>Relative residual stopping criterion</u>

$$\frac{\|f_0 - A_0 \tilde{y}_0\|}{\|f_0\|} \leq tol$$

Numerical experiment - relative residual stopping criterion

Problem

- 2D elliptic PDE, Poisson equation
- Find u: $-\Delta u = f$ in Ω, $u = g$ on $\partial\Omega$.



$$u(r,\theta) = r^{2/3}\sin\left(\frac{2}{3}\theta\right)$$

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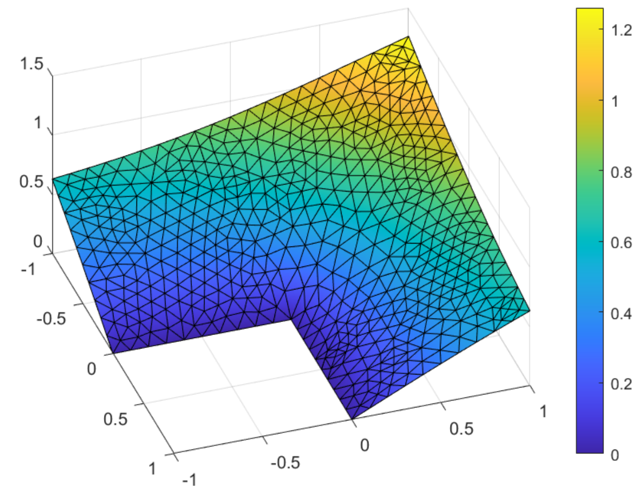

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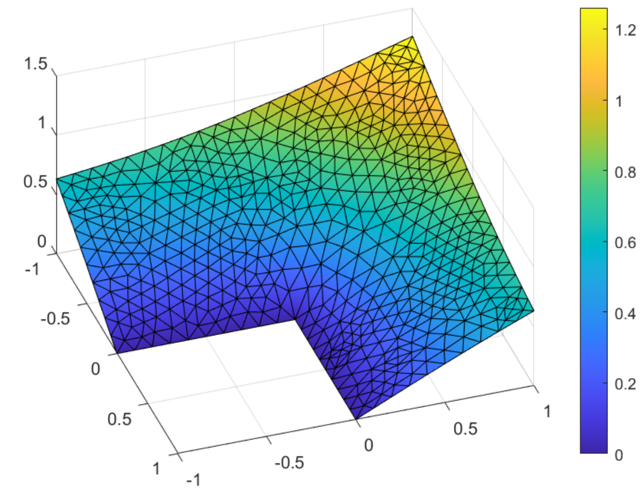
Discretization

- Galerkin FEM – continuous piecewise linear functions
- uniform refinement – 4 levels
- number of DOFs
  - coarsest level 22 849
  - finest level 1 477 281

Numerical experiment - relative residual stopping criterion



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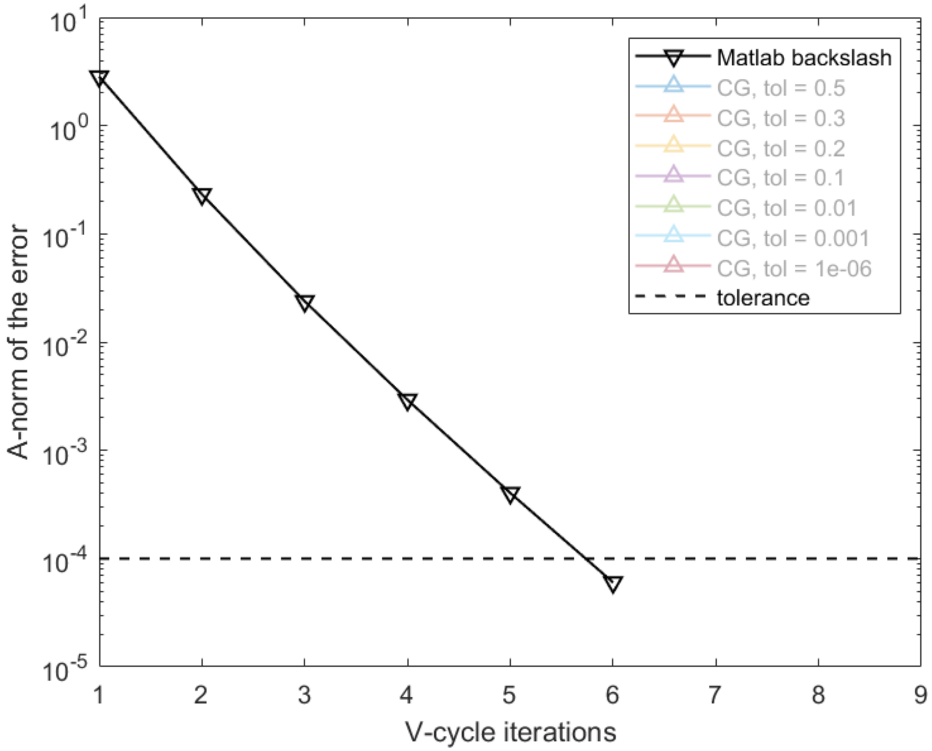
V-cycle method

- 4 levels
- smoothing – damped Jacobi method ($\omega = 2/3$)
- zero initial approximation

number of smoothing iterations		
level	pre	post
4	3	3
3	4	4
2	5	5

Numerical experiment - relative residual stopping criterion



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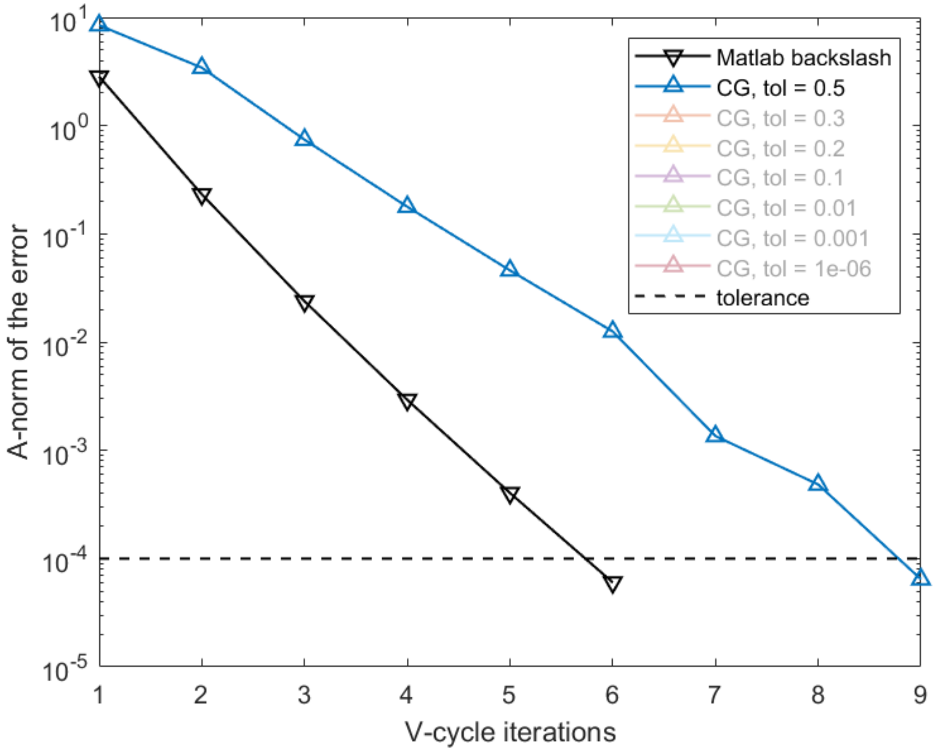

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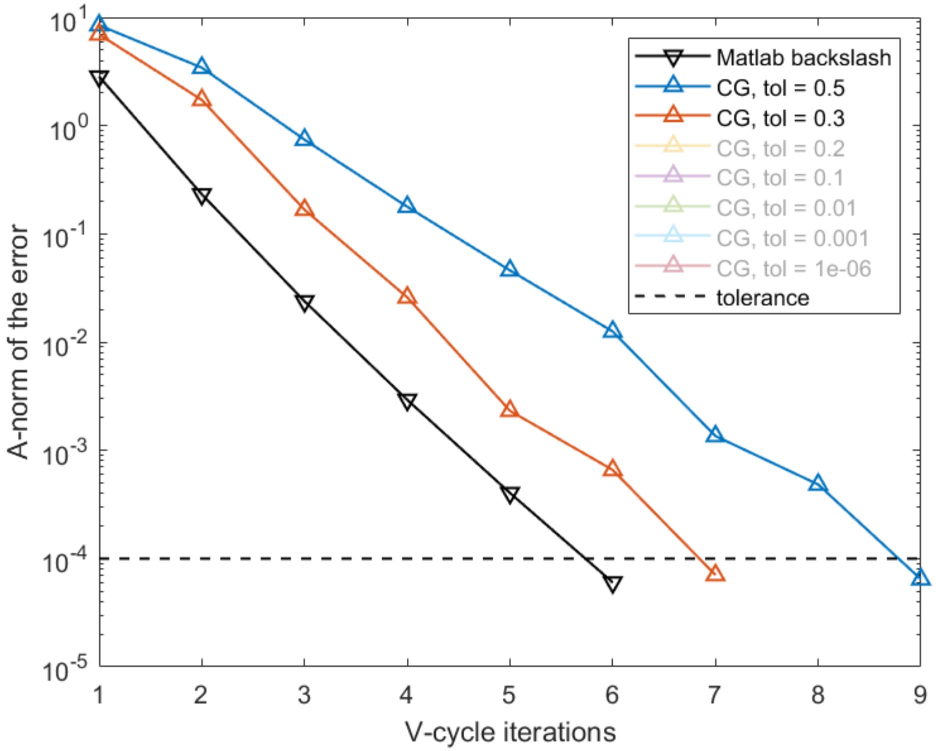

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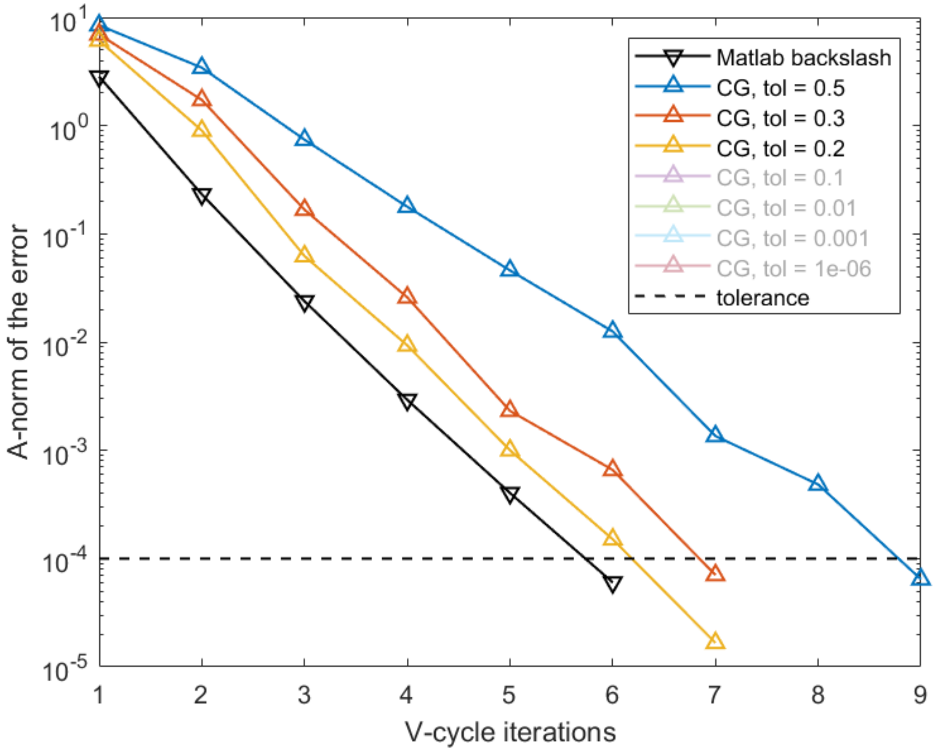

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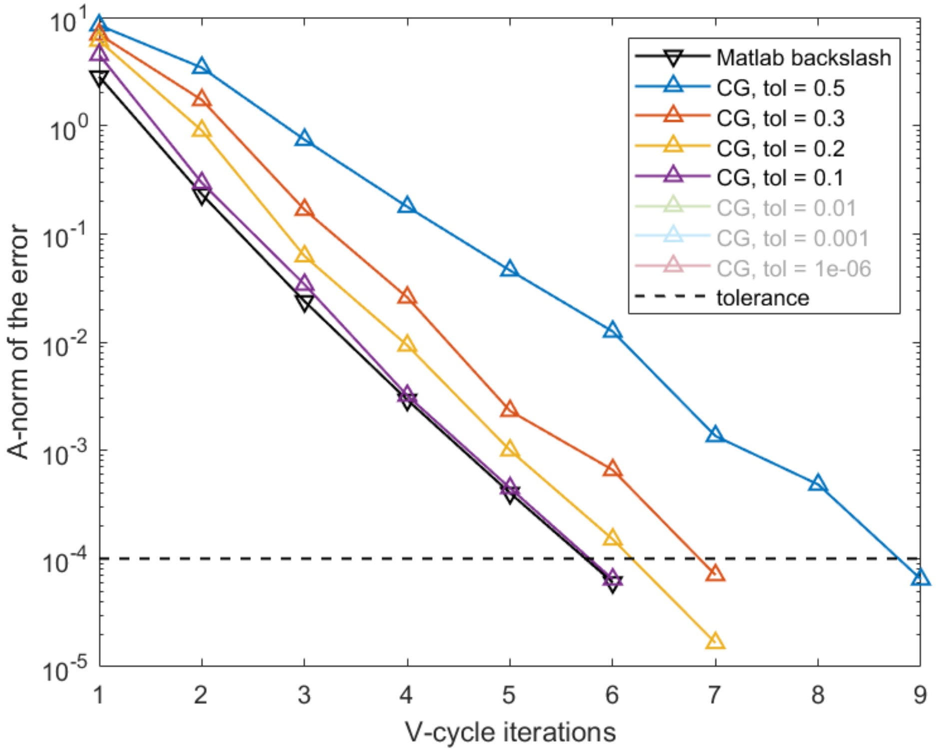

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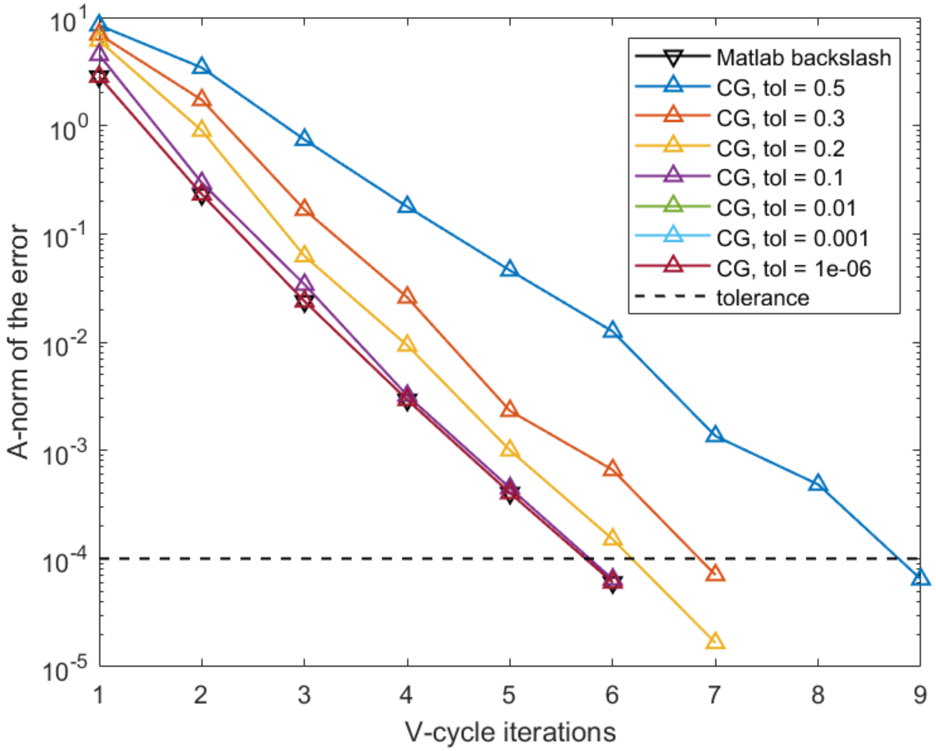

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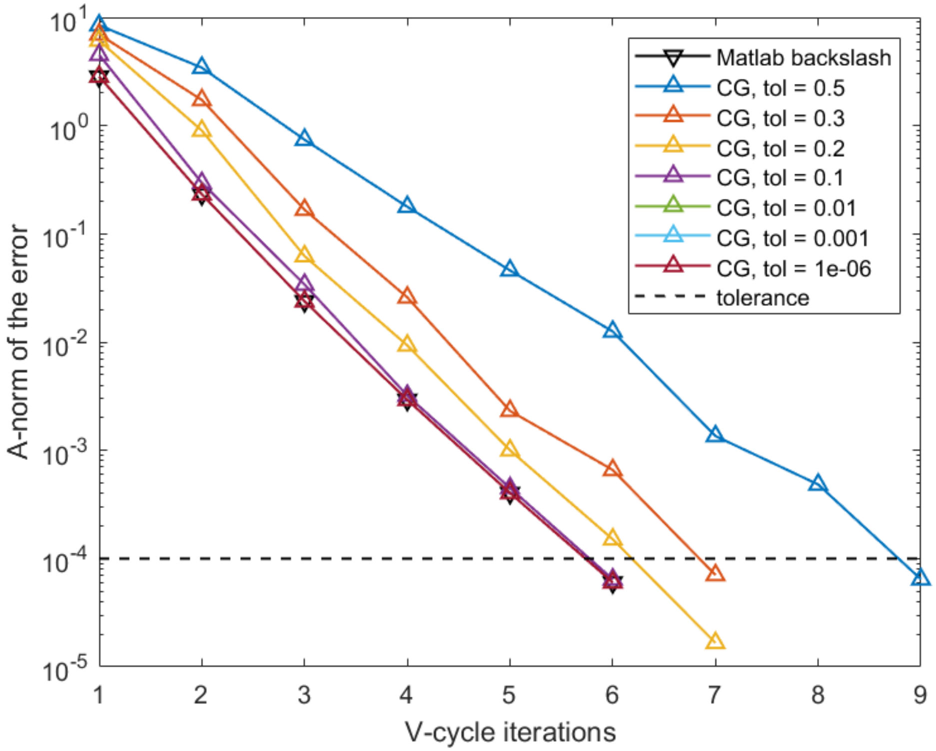

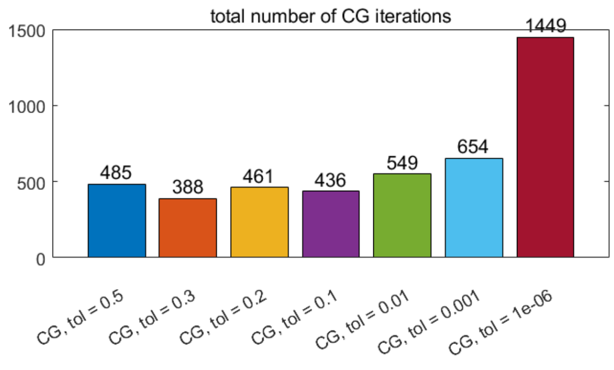

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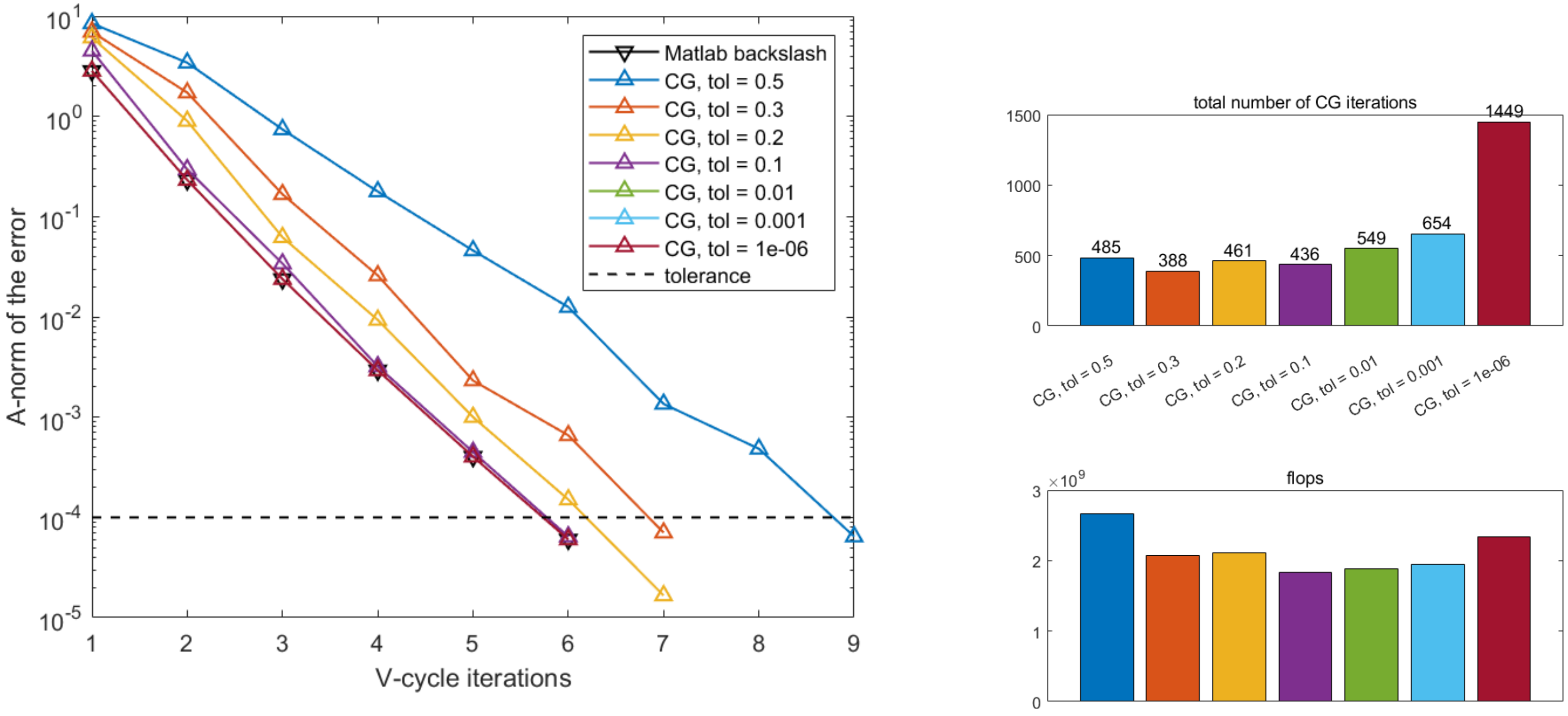

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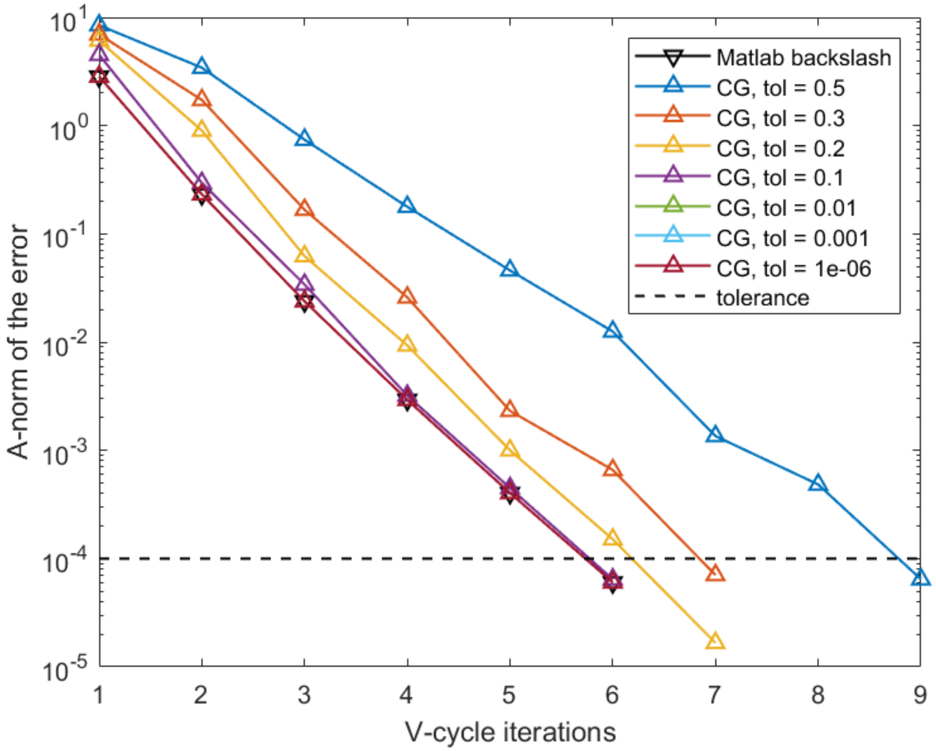

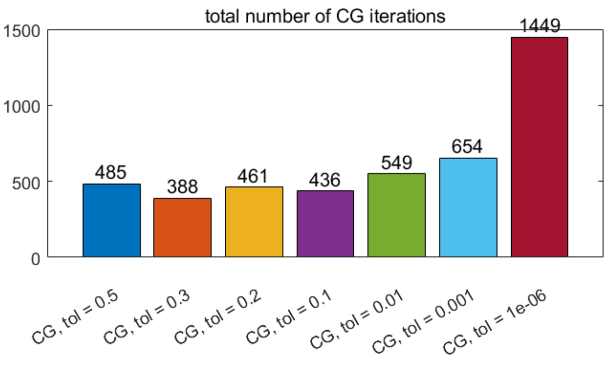

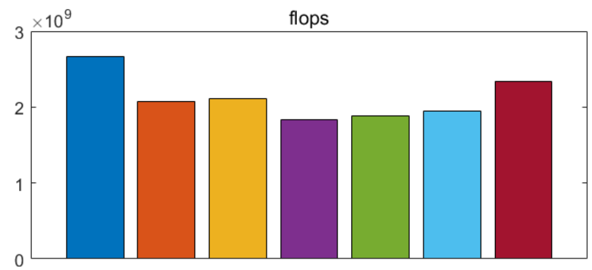

- Can we analytically describe how the relative tolerance affects the rate of convergence?

- Can we define a stopping criterion which would yield that the computed approximation is close to the one computed by V-cycle with exact solver and we do the fewest number of coarse grid iterations?

Delay in convergence rate after one V-cycle

<u>Setting</u>

- exact arithmetic
- A - symmetric positive definite matrix
- V-cycle method, $R_j = P_j^T$, $A_j = P_j^T A P_j$
- V-cycle with exact solver converges

Delay in convergence rate after one V-cycle

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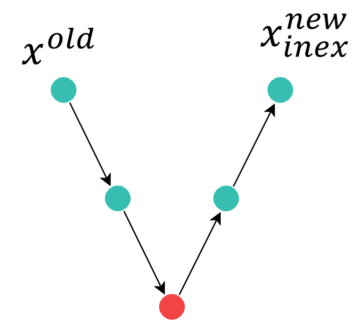


Find y_0: $A_0 y_0 = f_0$,

$\tilde{y}_0 \approx y_0$.

<u>Theorem</u>

Let x_{ex}^{new} be the approximation computed by V-cycle with exact solver.

Given $\gamma \in (0,1)$, let x_{inex}^{new} be an approximation computed by V-cycle with inexact solver, which is stopped when

$$\|y_0 - \tilde{y}_0\|_{A_0} \leq \gamma \cdot \left\|x - x^{old}\right\|_A .$$

Then

$$\frac{\|x - x_{inex}^{new}\|_A}{\left\|x - x^{old}\right\|_A} \leq \frac{\|x - x_{ex}^{new}\|_A}{\left\|x - x^{old}\right\|_A} + \gamma ,$$

$$\|x_{ex}^{new} - x_{inex}^{new}\|_A \leq \gamma \cdot \left\|x - x^{old}\right\|_A .$$

Delay in convergence rate after one V-cycle

Proof inspired by [van den Eshof and Sleijpen, 2004], [McCormick et al., 2020].

$$x_{inex}^{new} = x_{ex}^{new} - \left(\prod_{j=0}^{J-1} \left(I_{J-j} - M_{J-j}\, A_{J-j} \right) P_{J-j-1}^{J-j} \right) \left(y_0 - \tilde{y}_0 \right)$$

Delay in convergence rate after n V-cycles

<u>Theorem</u>

Let $x_{ex}^{(n)}$ be the approximation computed after n V-cycles with exact solver. Assume that the covergence rate of V-cycle with exact solver is bounded by $\rho \in (0,1)$, i.e.,

$$\frac{\|x - x_{ex}^{new}\|_A}{\|x - x^{old}\|_A} \leq \rho \qquad \forall x^{old}.$$

Given $\gamma \in (0,1)$ and let $x_{inex}^{(n)}$ be an approximation computed after n V-cycles with inexact solver, which is for all $k = 1, \dots, n$, stopped when

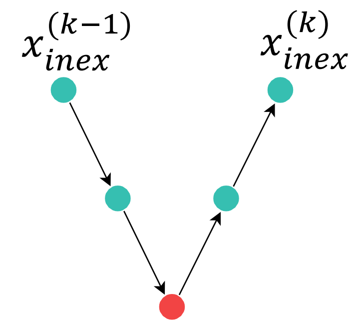
$$\left\| y_0^{(k)} - \tilde{y}_0^{(k)} \right\|_{A_0} \leq \gamma \cdot \left\| x - x_{inex}^{(k-1)} \right\|_A .$$

Then

$$\left\| x - x_{inex}^{(n)} \right\|_A \leq (\rho + \gamma)^n \left\| x - x^{(0)} \right\|_A ,$$

$$\left\| x_{ex}^{(n)} - x_{inex}^{(n)} \right\|_A \leq \left((\rho + \gamma)^n - \rho^n \right) \left\| x - x^{(0)} \right\|_A .$$

Find x : $\quad Ax = b$.



Find $y_0^{(k)}$: $A_0 y_0^{(k)} = f_0^{(k)}$,

$\tilde{y}_0^{(k)} \approx y_0^{(k)}$.

Residual based stopping criteria

Instead of $\|y_0 - \tilde{y}_0\|_{A_0} \leq \gamma \cdot \left\|x - x^{old}\right\|_A$, use estimates $\|y_0 - \tilde{y}_0\|_{A_0} \leq \alpha$ *and* $\beta \leq \left\|x - x^{old}\right\|_A$

and stop the solver when $\alpha \leq \gamma \cdot \beta$.

Then

$$\frac{\left\|x - x_{inex}^{new}\right\|_A}{\left\|x - x^{old}\right\|_A} \leq \frac{\|x - x_{ex}^{new}\|_A}{\left\|x - x^{old}\right\|_A} + \gamma.$$

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<u>1. Residual based</u>

$$\|y_0 - \tilde{y}_0\|_{A_0} \leq \sqrt{\|A_0^{-1}\|} \cdot \|f_0 - A_0\tilde{y}_0\| = \alpha$$

$$\beta = \frac{1}{\sqrt{\|A\|}} \cdot \left\|b - Ax^{old}\right\| \leq \left\|x - x^{old}\right\|_A$$

<u>Stop the solver when</u>

$$\|f_0 - A_0\tilde{y}_0\| \leq \frac{\gamma}{\sqrt{\|A_0^{-1}\| \cdot \|A\|}} \cdot \left\|b - Ax^{old}\right\|.$$

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<u>2. Relative residual</u>

$$\|y_0 - \tilde{y}_0\|_{A_0} \leq \sqrt{\|A_0^{-1}\| \cdot \|f_0 - A_0\tilde{y}_0\|} = \alpha$$

$$\beta = \frac{1}{\sqrt{\|A\| \cdot T(P,S)}} \cdot \|f_0\| \leq \left\|x - x^{old}\right\|_A$$

<u>Stop the solver when</u>

$$\frac{\|f_0 - A_0\tilde{y}_0\|}{\|f_0\|} \leq \frac{\gamma}{\sqrt{\|A_0^{-1}\| \cdot \|A\| \cdot T(P,S)}} = tol$$

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Then

$$\frac{\|x - x_{inex}^{new}\|_A}{\|x - x^{old}\|_A} \leq \frac{\|x - x_{ex}^{new}\|_A}{\|x - x^{old}\|_A} + \gamma.$$

<u>3. Multilevel error estimate + CG estimate (ML + CG)</u>

$$\|y_0 - \tilde{y}_0\|_{A_0} \leq \theta_{CG}(\tilde{y}_0) = \alpha$$

$$\beta = \eta_{ML}(x^{old}) \leq \|x - x^{old}\|_A$$

<u>Stop the solver when</u>

$$\theta_{CG}(\tilde{y}_0) \leq \gamma \cdot \eta_{ML}(x^{old})$$

Meurant, Papež and Tichý, Accurate error estimation in CG, Numerical Algorithms, Numerical Algorithms, Volume 88, pp. 1337-1359, 2021.

Rüde, Mathematical and computational techniques for multilevel adaptive methods, SIAM, Philadelphia, PA, 1993.

Efficiency of stopping criteria

$$Ef = \frac{\frac{\|x_{ex}^{new} - x_{inex}^{new}\|_A}{\|x - x^{old}\|_A}}{\gamma}$$

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Estimated efficiency for our numerical experiment	
Criterion with A-norms of the errors	0.95
1. Residual based	0.024
2. Relative residual	0.00076
3. ML + CG estimate	0.26

Heuristic strategy for choosing the convergence rate delay γ

$$\left\| x_{ex}^{(n)} - x_{inex}^{(n)} \right\|_A \leq \left((\rho + \gamma)^n - \rho^n \right) \left\| x - x^{(0)} \right\|_A$$

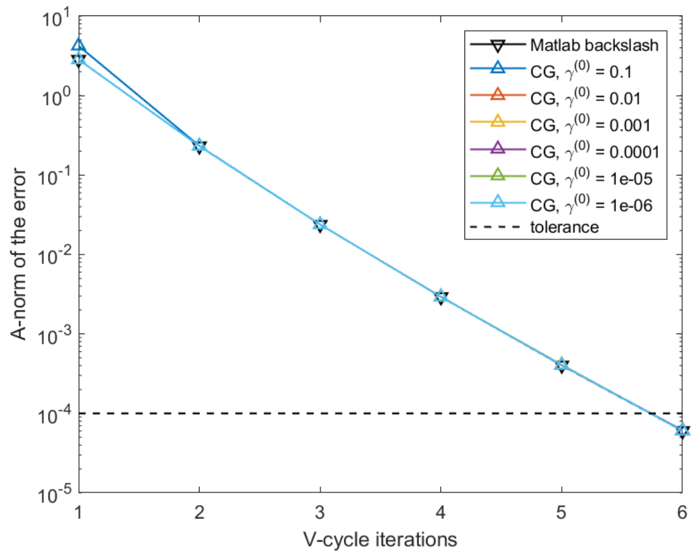
Choose initial $\gamma^{(0)}$ and N.

In each iteration:

$$\rho \approx \tilde{\rho}^{(k)} = \frac{\eta_{ML}^{(k-1)}}{\eta_{ML}^{(k-2)}}$$

$$\gamma^{(k)} = \left((2)^{\frac{1}{N}} - 1 \right) \tilde{\rho}^{(k)}$$

Heuristic strategy for choosing the convergence rate delay γ



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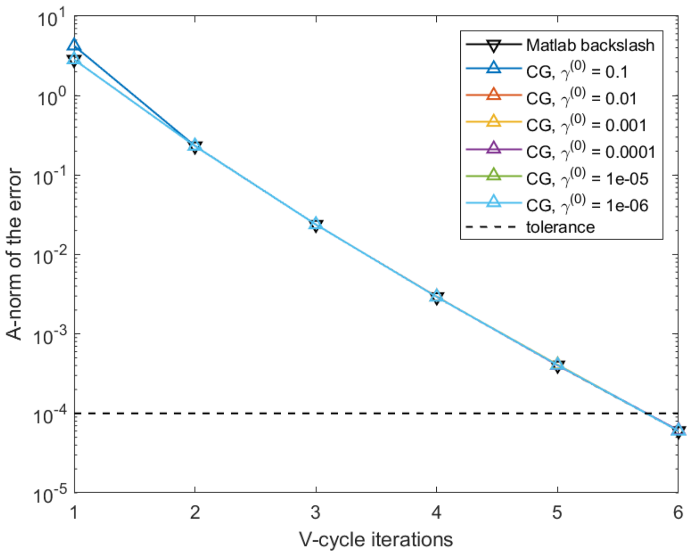

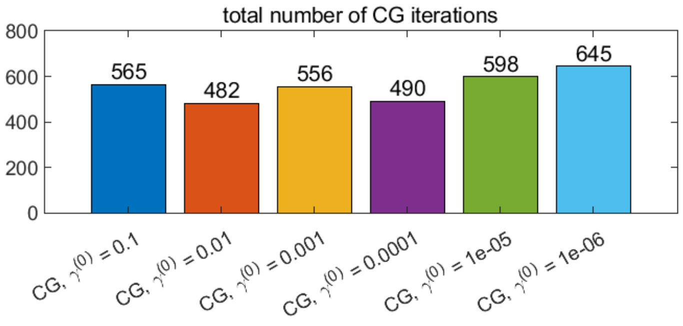

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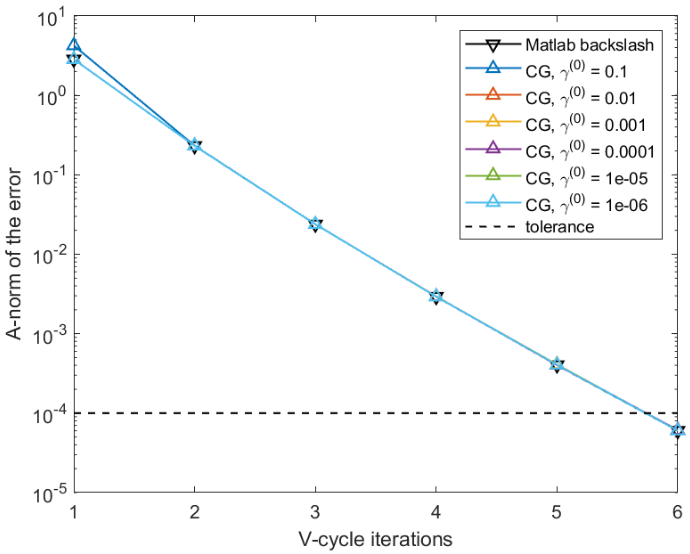
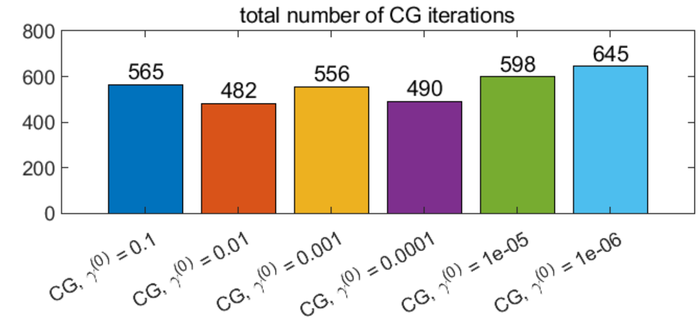
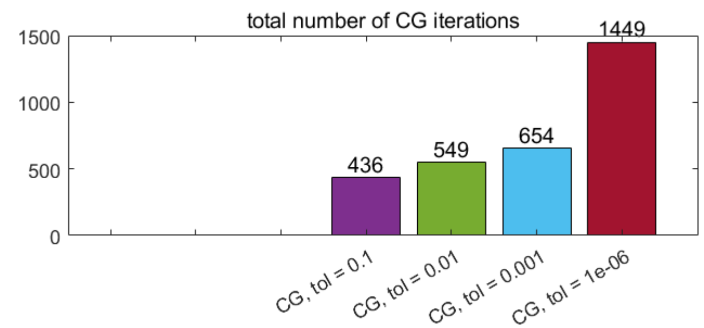




Relative tolerance



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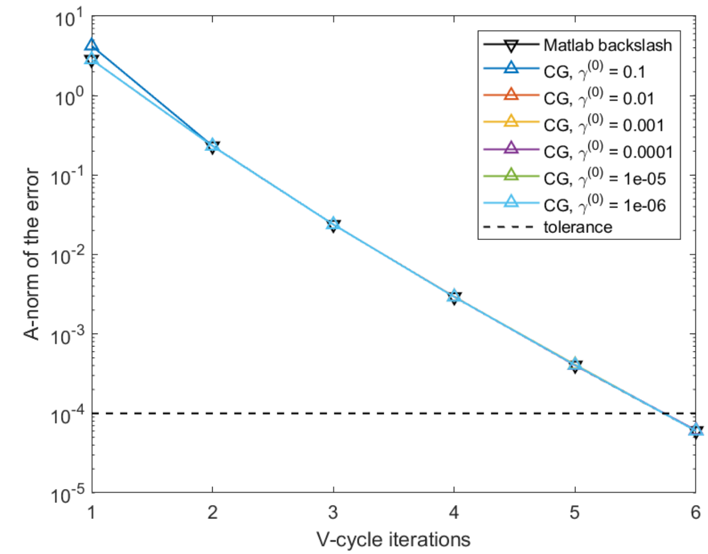

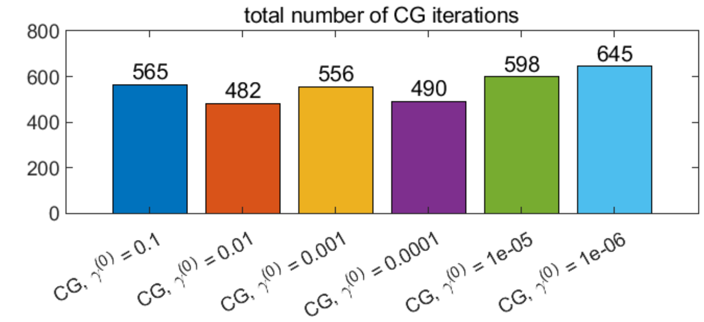

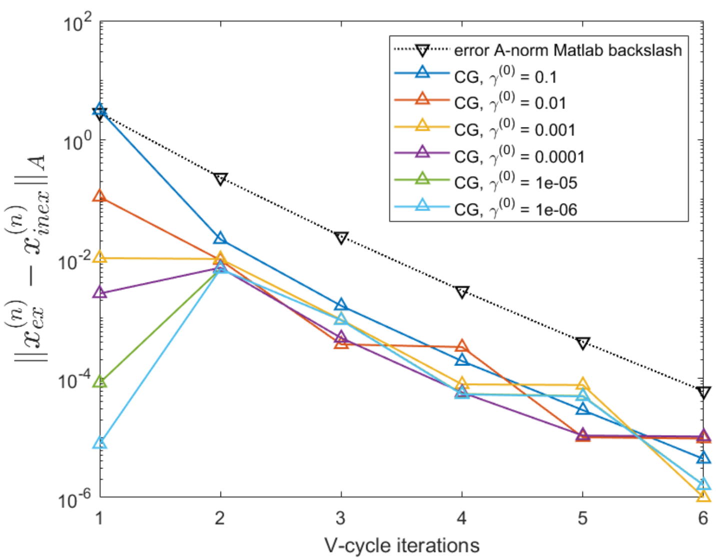

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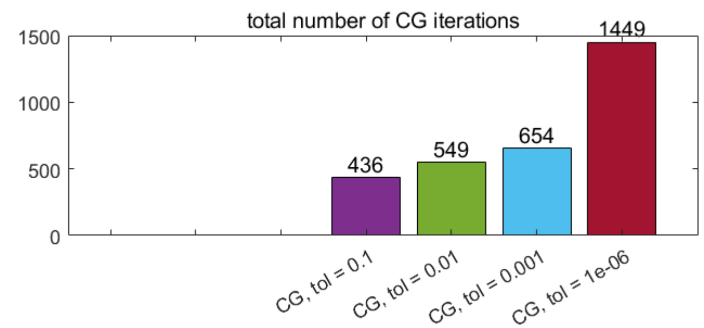

Thank you for your attention