Local projection stabilization for the numerical simulation of convection dominated flows

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joint work with

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Outline

– stabilization for incompressible flow problems

– local projection stabilization for the Oseen problem

– generalized formulation with overlapping projection domains

– stability and error analysis with respect to an improved norm

– optimal convergence results for correctly scaled stabilization parameters

Oseen problem

$$-\nu\,\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \sigma\,\mathbf{u} + \nabla p = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega\,,$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega$$

$\Omega \subset \mathbb{R}^d$, $d = 2, 3$ . . . bounded domain with a polyhedral Lipschitz–continuous boundary $\partial\Omega$

$\nu > 0$ and $\sigma \geq 0$ constants, $\mathbf{b} \in W^{1,\infty}(\Omega)^d$, $\mathbf{f} \in L^2(\Omega)^d$, $\operatorname{div}\mathbf{b} = 0$

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Weak formulation

Find $\mathbf{u} \in H^1_0(\Omega)^d$ and $p \in L^2_0(\Omega)$ such that

$$a(\mathbf{u},\mathbf{v}) - (p, \operatorname{div}\mathbf{v}) + (q, \operatorname{div}\mathbf{u}) = (\mathbf{f},\mathbf{v}) \quad \forall\, \mathbf{v} \in H^1_0(\Omega)^d,\, q \in L^2_0(\Omega)\,,$$

where

$$a(\mathbf{u},\mathbf{v}) = \nu\,(\nabla\mathbf{u},\nabla\mathbf{v}) + ((\mathbf{b}\cdot\nabla)\mathbf{u},\mathbf{v}) + \sigma\,(\mathbf{u},\mathbf{v})\,.$$

Galerkin discretization

Find $\mathbf{u}_h \in V_h^d$ and $p_h \in Q_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) + (q_h, \operatorname{div} \mathbf{u}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in V_h^d,\, q_h \in Q_h\,.$$

$V_h \subset H_0^1(\Omega)$, $Q_h \subset L_0^2(\Omega)$ ... finite–dimensional spaces

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Two sources of instabilities:

– dominant convection

– violation of the inf–sup condition

$$\sup_{\mathbf{v}_h \in V_h^d} \frac{(q_h, \operatorname{div} \mathbf{v}_h)}{|\mathbf{v}_h|_{1,\Omega}} \geq \beta\, \|q_h\|_{0,\Omega} \quad \forall\, q_h \in Q_h$$

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Residual-based stabilization (SUPG/PSPG/div-div)

Find $\mathbf{u}_h \in V_h^d$ and $p_h \in Q_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) + (q_h, \operatorname{div} \mathbf{u}_h)$$
$$+(-\nu\,\Delta_h \mathbf{u}_h + (\mathbf{b}\cdot\nabla)\mathbf{u}_h + \sigma\,\mathbf{u}_h + \nabla p_h - \mathbf{f}, \delta\,((\mathbf{b}\cdot\nabla)\mathbf{v}_h + \nabla q_h))$$
$$+(\operatorname{div} \mathbf{u}_h, \gamma \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in V_h^d,\, q_h \in Q_h\,.$$

Brooks, Hughes (1982)

Hughes, Franca, Balestra (1986)

Hansbo, Szepessy (1990)

Franca, Frey (1992)

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Drawbacks: non–symmetric, second–order derivatives,
difficulties for non–steady problems,
strong coupling between velocity and pressure

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Projection-based stabilization

$$\kappa_h = id - \pi_h$$

Find $\mathbf{u}_h \in V_h^d$ and $p_h \in Q_h$ such that

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$$+(\kappa_h((\mathbf{b} \cdot \nabla)\mathbf{u}_h), \delta^u \kappa_h((\mathbf{b} \cdot \nabla)\mathbf{v}_h)) + (\kappa_h \nabla p_h, \delta^p \kappa_h \nabla q_h)$$
$$+(\kappa_h \operatorname{div} \mathbf{u}_h, \gamma\, \kappa_h \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in V_h^d, q_h \in Q_h\,.$$

Codina (2000)

Kaya, Layton (2003)

Braack, Burman (2006)

Local projection stabilizations

Becker, Braack (2001) Stokes

Becker, Braack (2004) transport, Navier–Stokes

Braack, Burman (2006) Oseen

Braack, Richter (2006, 2007) Stokes; Navier–Stokes; react. flows

Becker, Vexler (2007) conv.–diff.–react., optimal control

Lube, Rapin, Löwe (2007) Oseen

Ganesan, Tobiska (2007) conv.–diff.–react., Stokes, Oseen

Matthies, Skrzypacz, Tobiska (2007) Oseen, enrichment

Matthies, Skrzypacz, Tobiska (2008) conv.–diff.–react.

Knobloch, Lube (2009) conv.–diff.–react.

Knobloch, Tobiska (2009) conv.–diff.–react.

Braack (2008, 2009) Navier–Stokes; Oseen, optimal control

Braack, Lube (2009) review on LPS for incompressible flows

Local projection stabilizations

Advantages: preserve the stability properties of RBS

no second order derivatives

no couplings between various unknowns

easy to apply to non–steady problems

symmetric

operations *discretization* and *optimization* commute Becker, Vexler (2007), Braack (2009)

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Drawbacks: more DOFs than RBS

in some cases less accurate

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One–level approach

Matthies, Skrzypacz, Tobiska (2007)

$\mathscr{M}_h = \mathscr{T}_h$

examples of spaces: $D_M = P_{l-1}(M) \qquad \forall\, M \in \mathscr{M}_h\,,$

$$V_h = P_{l,\mathscr{T}_h} \;+ \bigoplus_{M\in\mathscr{M}_h} b_M \cdot P_{l-1}(M)$$

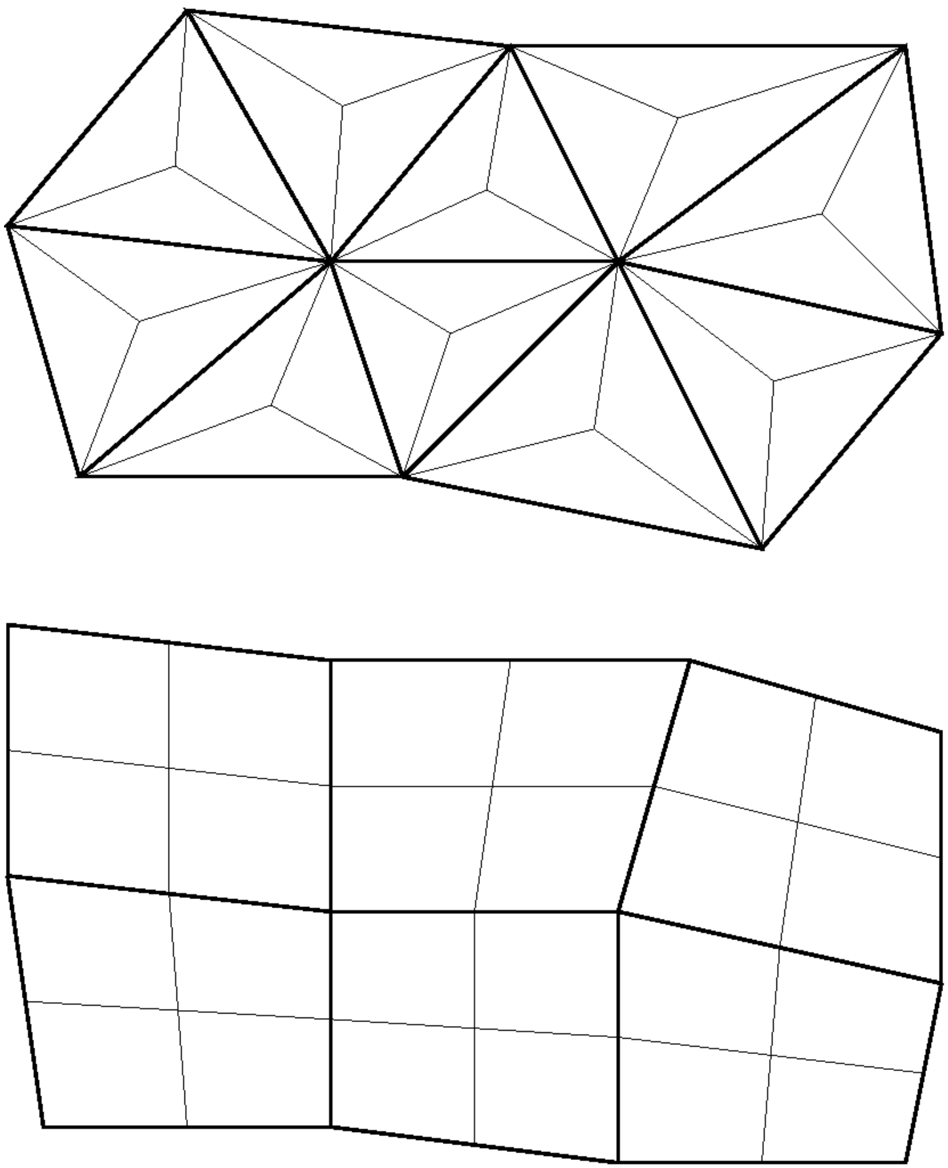
or $$V_h = Q_{l,\mathscr{T}_h} \;+ \bigoplus_{M\in\mathscr{M}_h} b_M \cdot Q_{l-1}(M)$$

(mapped)

Two–level approach

Becker, Braack (2001)

$\mathscr{T}_h$ is obtained by a refinement of $\mathscr{M}_h$



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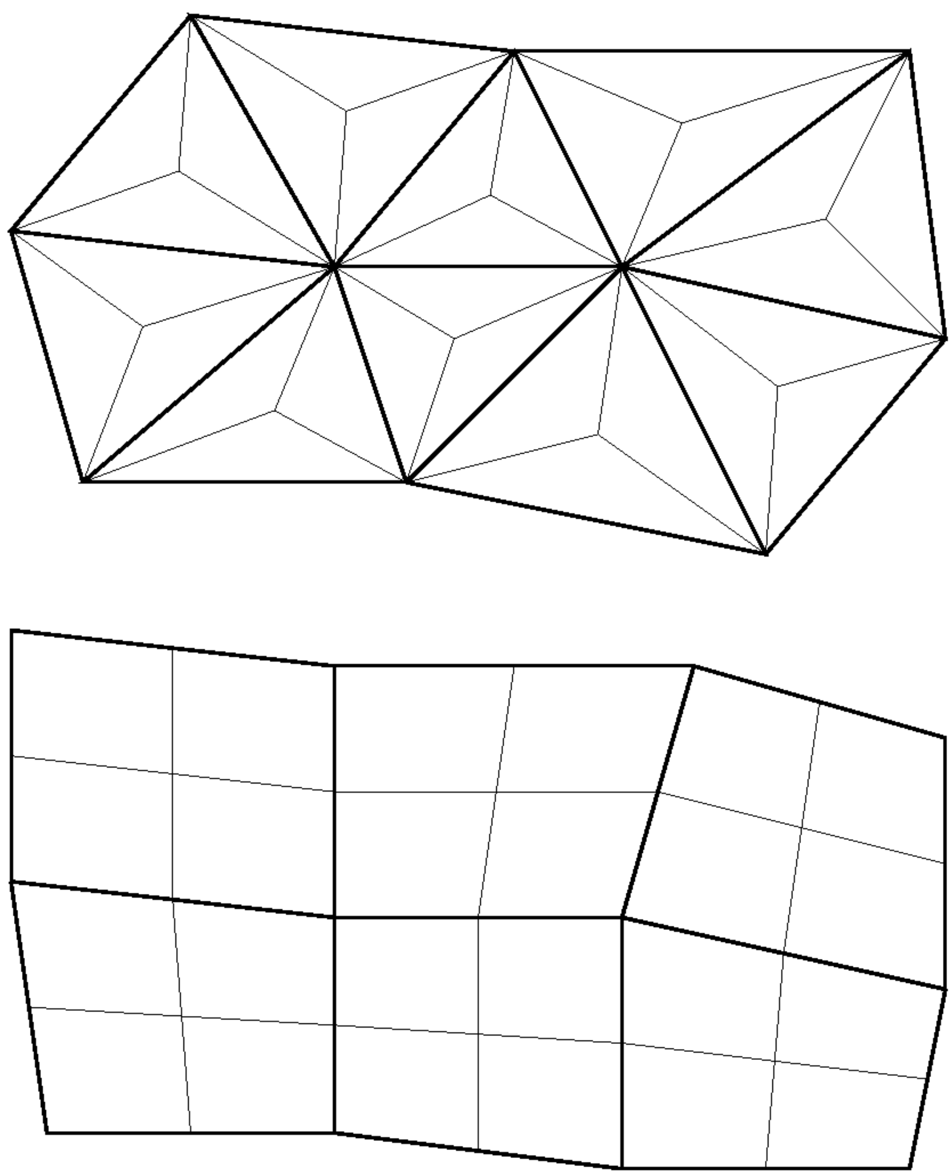

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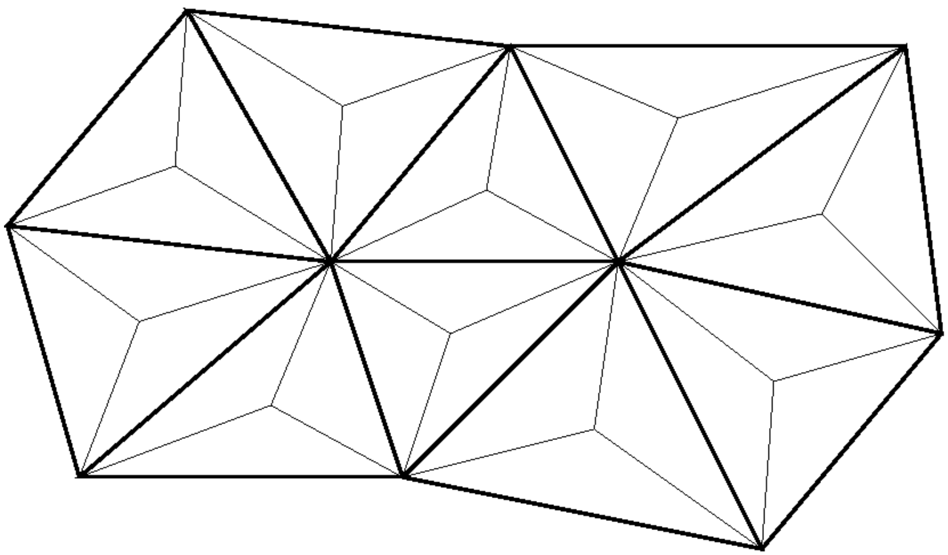

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can be viewed as one–level approach for simplicial meshes

Overlapping sets $M \in \mathscr{M}_h$ K. (2009)

Let any element of $\mathscr{T}_h$ have a vertex in Ω.

Let $x_1, \dots, x_{N_h}$ be the vertices of $\mathscr{T}_h$ lying in Ω.

Set
$$M_i = \operatorname{int} \bigcup_{T \in \mathscr{T}_h,\, x_i \in \overline{T}} \overline{T}\,, \qquad i = 1, \dots, N_h\,,$$

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cheaper and more robust than the previous approaches

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$\mathbf{b}_M \in \mathbb{R}^d$ such that

$$|\mathbf{b}_M| \le \|\mathbf{b}\|_{0,\infty,M}, \qquad \|\mathbf{b} - \mathbf{b}_M\|_{0,\infty,M} \le C\, h_M\, |\mathbf{b}|_{1,\infty,M}$$

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$Q_h \subset H^{1,h}(\Omega) \cap L_0^2(\Omega)$ ... FE space on $\mathscr{T}_h$

$$H^{1,h}(\Omega) = \{q \in L^2(\Omega)\,;\; q|_T \in H^1(T)\ \forall\, T \in \mathscr{T}_h\}$$

A local projection discretization

Find $\mathbf{u}_h \in V_h^d$ and $p_h \in Q_h$ such that

$$A_h([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in V_h^d,\, q_h \in Q_h\,,$$

where

$$\begin{aligned} A_h([\mathbf{u}, p], [\mathbf{v}, q]) &= a(\mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) + (q, \operatorname{div} \mathbf{u}) \\ &\quad + s_h^b(\mathbf{u}, \mathbf{v}) + s_h^u(\mathbf{u}, \mathbf{v}) + s_h^p(p, q) + s_h^j(p, q)\,. \end{aligned}$$

A local projection discretization

$$s_h^b(\mathbf{u},\mathbf{v}) = \sum_{M\in\mathscr{M}_h} \tau_M \left(\kappa_M[(\mathbf{b}_M\cdot\nabla)\mathbf{u}], \kappa_M[(\mathbf{b}_M\cdot\nabla)\mathbf{v}]\right)_M ,$$

$$s_h^u(\mathbf{u},\mathbf{v}) = \sum_{M\in\mathscr{M}_h} \mu_M \left(\kappa_M(\operatorname{div}\mathbf{u}), \kappa_M(\operatorname{div}\mathbf{v})\right)_M ,$$

$$s_h^p(p,q) = \sum_{M\in\mathscr{M}_h} \alpha_M \left(\kappa_M(\nabla_h p), \kappa_M(\nabla_h q)\right)_M ,$$

$$s_h^j(p,q) = \sum_{E\in\mathscr{E}_h} \beta_E \left([p]_E, [q]_E\right)_E$$

Stabilization parameters:

$$\tau_M \approx \gamma_M := \frac{h_M^2}{\nu + h_M \|\mathbf{b}\|_{0,\infty,M} + h_M^2\,\sigma} ,$$

$$\mu_M \approx \nu + h_M^2\,\sigma , \qquad \alpha_M \approx \frac{h_M^2}{\nu + h_M^2\,\sigma} , \qquad \beta_E \approx \frac{h_E}{\nu + h_E^2\,\sigma}$$

(or $\mu_M = 0$)

Stability of the local projection discretization

Local projection norm:

$$|||[\mathbf{v},q]|||_{LP} = \Big(\nu\,|\mathbf{v}|_{1,\Omega}^2 + \sigma\,\|\mathbf{v}\|_{0,\Omega}^2 + s_h^b(\mathbf{v},\mathbf{v}) + s_h^u(\mathbf{v},\mathbf{v}) + s_h^p(q,q) + s_h^j(q,q)\Big)^{1/2}$$

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Stronger norm:

$$|||[\mathbf{v},q]||| = \left(|||[\mathbf{v},q]|||^2_{LP} + \frac{1}{1+\omega_h^1}\sum_{M\in\mathscr{M}_h}\gamma_M\,\|(\mathbf{b}\cdot\nabla)\mathbf{v} + \nabla_h q\|^2_{0,M}\right)^{1/2}$$

with

$$\omega_h^1 = \max_{M\in\mathscr{M}_h}\frac{h_M^2\,|\mathbf{b}|_{1,\infty,M}}{\nu + h_M^2\,\sigma}$$

Stability of the local projection discretization

Theorem $\exists \beta > 0$ such that, for any $\mathbf{u}_h \in V_h^d$ and $p_h \in Q_h$,

$$\sup_{[\mathbf{v}_h, q_h] \in V_h^d \times Q_h} \frac{A_h([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h])}{|||[\mathbf{v}_h, q_h]|||} \geq \beta \, |||[\mathbf{u}_h, p_h]||| \, .$$

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General error estimate for any $\mathbf{w}_h \in V_h^d$ and $r_h \in Q_h$

$$\begin{aligned} \beta \, |||[\mathbf{u} - \mathbf{u}_h, p - p_h]||| \leq & \; \beta \, |||[\mathbf{u} - \mathbf{w}_h, p - r_h]||| \\ & + \sup_{[\mathbf{v}_h, q_h] \in V_h^d \times Q_h} \frac{A_h([\mathbf{u} - \mathbf{w}_h, p - r_h], [\mathbf{v}_h, q_h])}{|||[\mathbf{v}_h, q_h]|||} \\ & + \sup_{[\mathbf{v}_h, q_h] \in V_h^d \times Q_h} \frac{s_h^b(\mathbf{u}, \mathbf{v}_h) + s_h^p(p, q_h)}{|||[\mathbf{v}_h, q_h]|||} \end{aligned}$$

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For an optimal estimate of the consistency error, it is essential that we use $\mathbf{b}_M$ instead of $\mathbf{b}$ in s_h^b.

Approximation properties of the spaces V_h, Q_h and D_M

$\exists\ i_h \in \mathscr{L}(H^2(\Omega)\cap H_0^1(\Omega), V_h)$ and $j_h \in \mathscr{L}(H^1(\Omega)\cap L_0^2(\Omega), Q_h)$ such that, for some constants $l_V \in \mathbb{N}$, $l_Q \in \mathbb{N}_0$ and C and for any $M \in \mathscr{M}_h$,

$$|v - i_h v|_{1,M} + h_M^{-1}\,\|v - i_h v\|_{0,M} \le C\,h_M^r\,|v|_{r+1,M} \qquad \forall\, v \in H^{r+1}(\Omega),\ r = 1,\dots,l_V\,,$$

$$\|\nabla_h(v - j_h v)\|_{0,M} + h_M^{-1}\,\|v - j_h v\|_{0,M} \le C\,h_M^r\,|v|_{r+1,M} \qquad \forall\, v \in H^{r+1}(\Omega)\cap L_0^2(\Omega),\ r = 0,\dots,l_Q\,.$$

$\exists$ constants $l_D \in \mathbb{N}$ and C such that

$$\inf_{v\in D_M} \|q - v\|_{0,M} \le C\,h_M^r\,|q|_{r,M} \qquad \forall\, q \in H^r(M),\ M \in \mathscr{M}_h,\ r = 1,\dots,l_D\,.$$

A priori error estimates

Theorem Let $\mathbf{u} \in H^{r+1}(\Omega)^d$ and $p \in H^{s+1}(\Omega)$ with $1 \leq r \leq \min\{l_V, l_D\}$ and $0 \leq s \leq \min\{l_Q, l_D\}$. Then

$$|||[\mathbf{u}-\mathbf{u}_h, p-p_h]||| \leq C\,h^r\,(1+\omega_h^1)^{1/2}\left(\sum_{M\in\mathscr{M}_h}\delta_M\,|\mathbf{u}|^2_{r+1,M}\right)^{1/2} + C\,h^s\left(\sum_{M\in\mathscr{M}_h}\alpha_M\,|p|^2_{s+1,M}\right)^{1/2},$$

where

$$\delta_M = \nu + h_M\,\|\mathbf{b}\|_{0,\infty,M} + h_M^2\,\sigma\,, \qquad \alpha_M \approx \frac{h_M^2}{\nu + h_M^2\,\sigma}$$

and C is independent of h and the data.

Estimate of $\|p - p_h\|_{0,\Omega}$

Lemma There is a constant $\gamma > 0$ independent of h such that, for any $q \in H^{1,h}(\Omega) \cap L_0^2(\Omega)$,

$$\sup_{\mathbf{v}_h \in V_h^d} \frac{(q, \operatorname{div} \mathbf{v}_h)}{|\mathbf{v}_h|_{1,\Omega}} + \left(\sum_{M \in \mathscr{M}_h} h_M^2 \, \|\kappa_M \nabla_h q\|_{0,M}^2 \right)^{1/2} + \left(\sum_{E \in \mathscr{E}_h} h_E \, \|[q]_E\|_{0,E}^2 \right)^{1/2} \geq \gamma \|q\|_{0,\Omega}.$$

Estimate of $\|p - p_h\|_{0,\Omega}$

Theorem Let $\mathbf{u} \in H^{r+1}(\Omega)^d$ with $r \in \{0, \ldots, l_D\}$ and $p \in H^1(\Omega)$. Then

$$\begin{aligned}
&\|p - p_h\|_{0,\Omega} \\
&\quad \leq C\left(\nu + C_F^2\,\sigma\right)^{1/2} \left(1 + \frac{C_F\,\|\mathbf{b}\|_{0,\infty,\Omega}}{\nu + C_F^2\,\sigma}\right) |||[\mathbf{u} - \mathbf{u}_h, p - p_h]|||_{LP} \\
&\qquad + C\left(\nu + h\,\|\mathbf{b}\|_{0,\infty,\Omega} + C_F^2\,\sigma\right)^{1/2} h^{r+1/2}\,\|\mathbf{b}\|_{0,\infty,\Omega}^{1/2}\,|\mathbf{u}|_{r+1,\Omega}\,,
\end{aligned}$$

where C is independent of h and the data.

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$\equiv$ Oseen problem with $\nu = 1$, $\mathbf{b} = \mathbf{0}$, $\sigma = 0$

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Theorem Let $\mathbf{u} \in H^{r+1}(\Omega)^d$ and $p \in H^{s+1}(\Omega)$ with $1 \leq r \leq l_V$ and $0 \leq s \leq \min\{l_Q, l_D\}$. Then

$$|||[\mathbf{u} - \mathbf{u}_h, p - p_h]||| + \|p - p_h\|_{0,\Omega} \leq C\,h^r\,|\mathbf{u}|_{r+1,\Omega} + C\,h^{s+1}\,|p|_{s+1,\Omega}\,.$$

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