5.3 Two numeric covariates

Now: two covariates Z, W

Z numeric W: numeric

Example: $=$ engine size $=$ weight

Cars2004nh

$Y =$ consumption

To start: Both Z and W parameterized by simple transformation

$\rightarrow$ $S_Z(z) \in \mathbb{R}^1$

$S_W(w) \in \mathbb{R}^1$

Example

$S_Z(z) = z$

$S_W(w) = \log(w)$

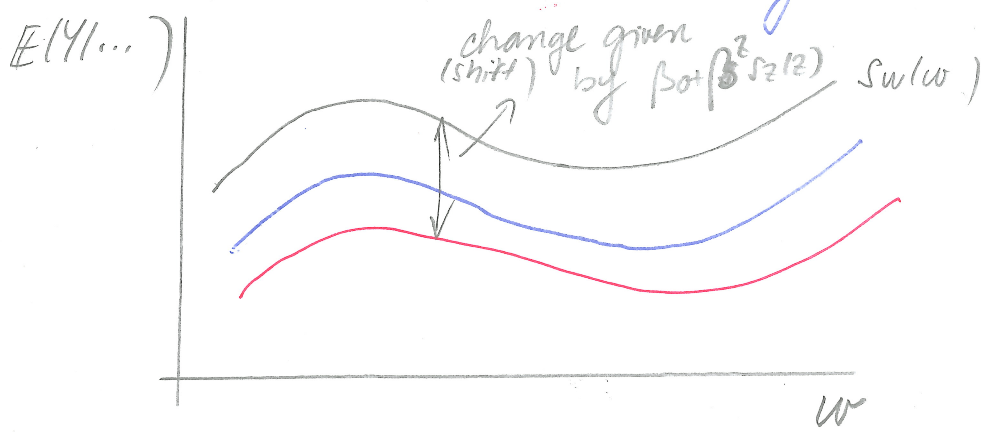
5.3.1 Additivity

MODEL:
$$E(Y|Z=z, W=w) = m(z,w)$$
$$= \beta_0 + \beta^z \cdot s_z(z) + \beta^w s_w(w)$$

Interpretation:

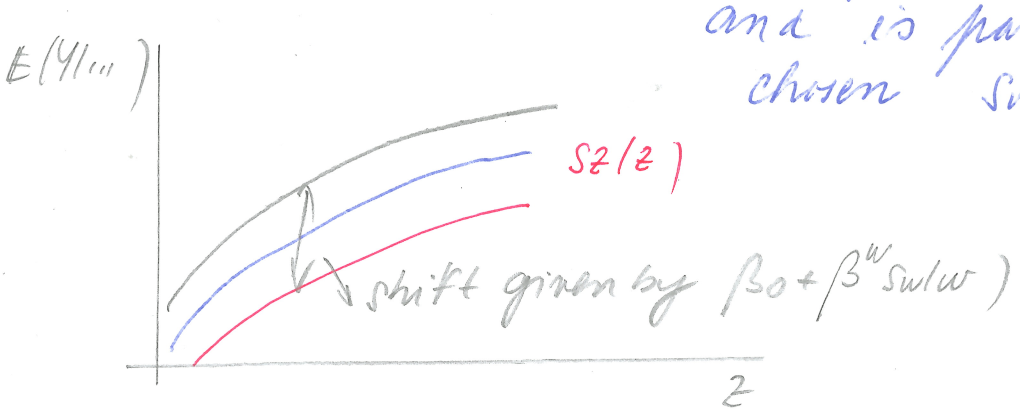
$$E(Y|Z=z, W=w) = \underbrace{\beta_0 + \beta^z s_z(z)} + \beta^w s_w(w)$$

interrept which depends on z and is parameterized by chosen s_z



$$E(Y|Z=z, W=w) = \underbrace{\beta_0 + \beta^w s_w(w)} + \beta^z s_z(z)$$

intercept which depends on w and is parameterized by chosen s_w

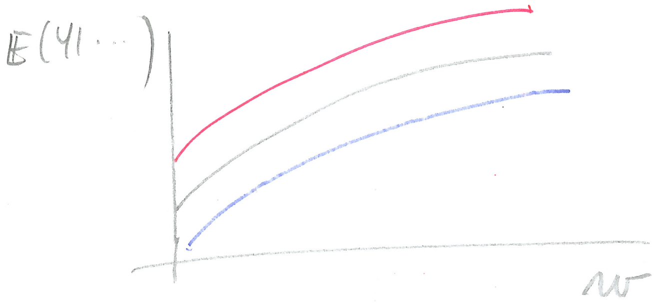


Example

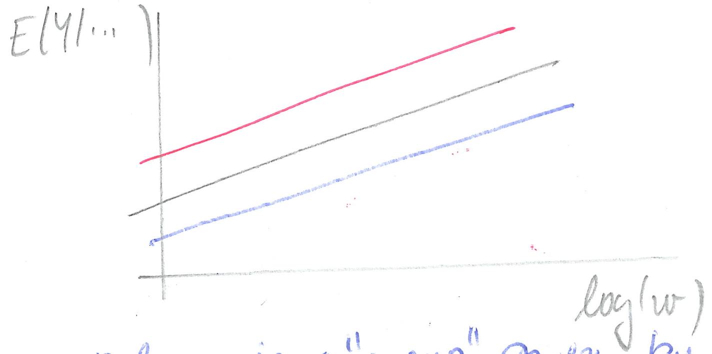
28

$$\mathbb{E}(Y \mid Z=z, W=w) = \beta_0 + \beta^z z + \beta^w \log(w)$$

consumption — engine size — weight



29



Params. of line in a "group" given by $Z=z$:

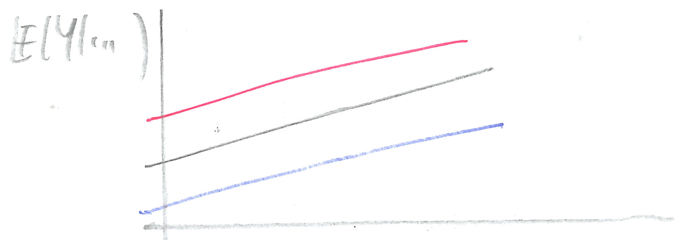
30

$$\text{intercept}(z) = \beta_0 + \beta^z \cdot z$$

$$\text{slope}(z) = \beta^w$$ ← does not depend on z (additivity assumed)

$$\mathbb{E}(Y \mid Z=z, W=w) = \beta_0 + \beta^w \log(w) + \beta^z z$$

31



Params. of line in a "group" z given by $W=w$:

32

$$\text{intercept}(w) = \beta_0 + \beta^w \log(w)$$

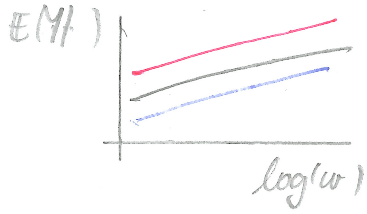
$$\text{slope}(w) = \beta^z$$ ← does not depend on w

5.3.2 Partial effects

33

Additivity: Effect of Z (or W) on the response expectation does not depend on a value of W (or Z).

MODEL: $E(Y / Z=z, W=w) = \beta_0 + \beta^z s_z(z) + \beta^w s_w(w)$



34

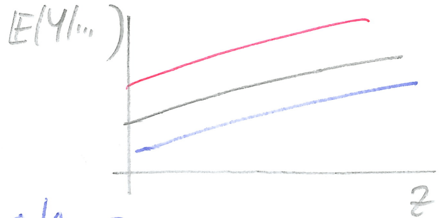
Partial effect of covariate w:

$E(Y | Z=z, W=w+1) - E(Y | Z=z, W=w) =$

$= \beta^w (s_w(w+1) - s_w(w))$

No effect $\equiv H_0: \beta^w = 0$ 35

$\to$ t-test on regress. coef.



36

Partial effect of covariate z:

$E(Y / Z=z+1, W=w) - E(Y / Z=z, W=w) =$

$= \beta^z (s_z(z+1) - s_z(z))$

No effect $\equiv H_0: \beta^z = 0$ 37

$\to$ t-test on regress. coef.

5.3.3 Interactions

38

$$s_W^T(w) \otimes s_Z^T(z) = \underset{1}{s_W(w)} \cdot \underset{1}{s_Z(z)} =: s_{ZW}(z,w)$$

$$\underset{1}{s_W(w)} \otimes \underset{1}{s_Z(z)}$$

<u>Interaction model</u> in general

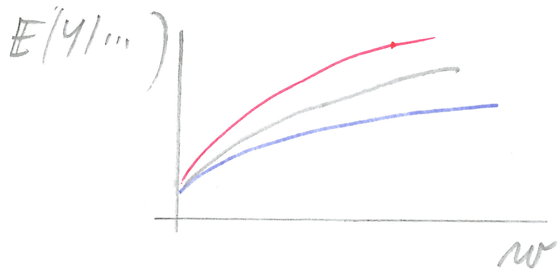
$$\mathbb{E}(Y|Z=z, W=w) = \beta_0 + s_Z^T(z)\beta^Z + s_W^T(w)\beta^W + s_{ZW}^T(z,w)\beta^{ZW}$$

$$\overset{\text{now}}{=} \beta_0 + \beta^Z s_Z(z) + \beta^W s_W(w) + \beta^{ZW} s_Z(z)\cdot s_W(w)$$

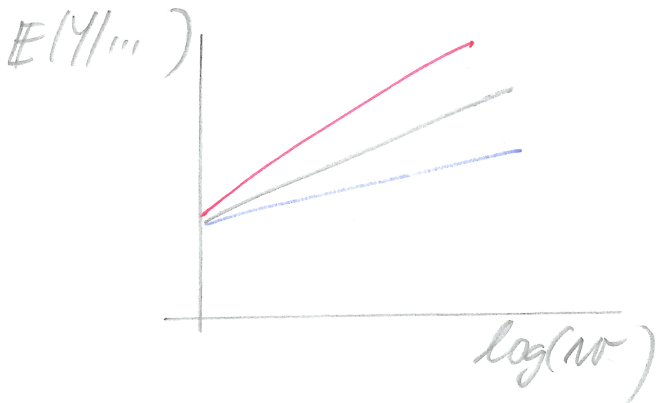
$$= \underbrace{\left(\beta_0 + \beta^Z s_Z(z)\right)}_{\gamma_0^W(z)} + \underbrace{\left(\beta^W + \beta^{ZW} s_Z(z)\right)}_{\gamma_1^W(z)} s_W(w)$$

intercept depends on z <u>linearly</u> through $s_Z(z)$

slope depends on z <u>linearly</u> through $s_Z(z)$



39



40

Interaction model in general (repetition):

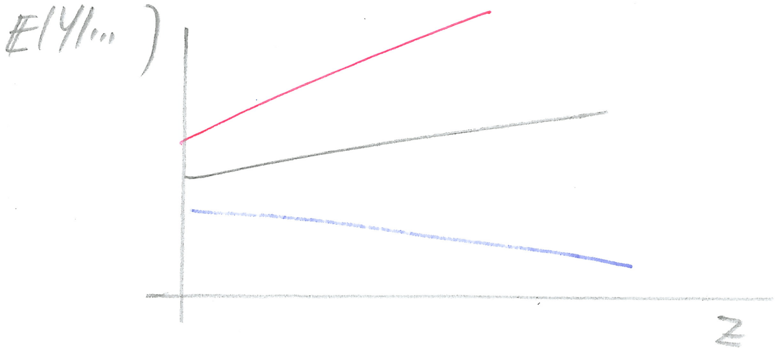
$$E(Y \mid Z=z, W=w) = \beta_0 + s_Z^T(z)\beta^Z + s_W^T(w)\beta^W + s_{ZW}^T(z,w)\beta^{ZW}$$

$$\overset{\text{NOW}}{=} \beta_0 + \beta^Z s_Z(z) + \beta^W s_W(w) + \beta^{ZW} s_Z(z) \cdot s_W(w)$$

$$= \underbrace{\left(\beta_0 + \beta^W s_W(w)\right)}_{\gamma_0^Z(w)} + \underbrace{\left(\beta^Z + \beta^{ZW} s_W(w)\right)}_{\gamma_1^Z(w)} s_Z(z)$$

intercept depends on w linearly through $s_W(w)$

slope depends on w linearly through $s_W(w)$



R output

- brief discussion on estimated β's

5.3.4 Additivity or interactions? [43]

Can additivity be assumed?

ASSUMED
$$E(Y \mid Z=z, W=w) = \beta_0 + \beta^z s_z(z) + \beta^w s_w(w) + \beta^{zw} s_z(z) \cdot s_w(w)$$

$$\longrightarrow \quad \beta_0 + \beta^z s_z(z) + \beta^w s_w(w)$$ [44-46] plots

$H_0: \beta^{zw} = 0$ (in interaction model)

→ t-test on regression coefficient (or equivalently submodel F-test)

t-test on regress. coef [47]

F-test on submodel [48]

5.3.5 More complex parameterization of either covariate

- either of the two covariates can (indeed) be parameterized in a more complex way using

$$\mathcal{S}_Z = (S_Z^1, \ldots, S_Z^{k-1})^T$$

$$\mathcal{S}_W = (S_W^1, \ldots, S_W^{l-1})^T$$

→ polynomials

→ splines

<u>Additivity model</u>

$$E(Y \mid Z = z, W = w) = \underbrace{\beta_0 + \mathcal{S}_Z^T(z)\beta^Z}_{} + \mathcal{S}_W^T(w)\beta^W$$

intercept $\gamma_0^W(z)$ which depends on z <u>linearly</u> through $\mathcal{S}_Z(z)$)

$$= \underbrace{\beta_0 + \mathcal{S}_W^T(w)\beta^W}_{} + \mathcal{S}_Z^T(z)\beta^Z$$

intercept $\gamma_0^Z(w)$ which depends on w <u>linearly</u> through $\mathcal{S}_W(w)$)

Interaction model

$$E(Y \mid Z=z, W=w) = \beta_0 + \mathcal{S}_Z^T(z)\beta^Z + \mathcal{S}_W^T(w)\underbrace{\beta^W}_{=(\beta_1^W,\dots,\beta_{l-1}^W)^T} +$$

$$+ \underbrace{\left(\mathcal{S}_W^T(w) \otimes \mathcal{S}_Z^T(z)\right)}_{\left(S_W^1(w)\,\mathcal{S}_Z^T(z),\,\dots,\,S_W^{l-1}(w)\,\mathcal{S}_Z^T(z)\right)} \beta^{ZW}$$

$$\beta^{ZW} = (\underbrace{\beta_{11}^{ZW},\dots,\beta_{k-1,1}^{ZW}}_{\beta_{\cdot,1}^{ZW}},\dots\dots,\underbrace{\beta_{1,l-1}^{ZW},\dots,\beta_{k-1,l-1}^{ZW}}_{\beta_{\cdot,l-1}^{ZW}})^T$$

$$= \beta_0 + \mathcal{S}_Z^T(z)\beta^Z + \sum_{j=1}^{l-1}\beta_j^W s_W^j(w) + \sum_{j=1}^{l-1}\mathcal{S}_Z^T(z)\beta_{\cdot j}^{ZW}\cdot S_W^j(w) =$$

$$= \underbrace{\left(\beta_0 + \mathcal{S}_Z^T(z)\beta^Z\right)}_{\mu_0^W(z)} + \sum_{j=1}^{l-1}\underbrace{\left(\beta_j^W + \mathcal{S}_Z^T(z)\beta_{\cdot j}^{ZW}\right)}_{\mu_j^W(z)} S_W^j(w)$$

$\mu_0^W(z)$ = intercepts which depend on z linearly through $\mathcal{S}_Z(z)$

$\mu_j^W(z)$ = slopes which depend on z linearly through $\mathcal{S}_Z(z)$)

---

e.g. $\mathcal{S}_Z(z) = z$, $S_W(w) = (w, w^2, \dots, w^{l-1})^T$ polynomial of degree $l-1$

$$E(Y \mid Z=z, W=w) = (\beta_0 + \beta^Z\cdot z) + \sum_{j=1}^{l-1}\underbrace{\left(\beta_j^W + \beta_j^{ZW}\cdot z\right)}_{\text{polynomial coefs. which depend linearly on } z} w^j$$

polynomial in w

- role of Z and W can (indeed) be reversed

e.g. $\delta_Z(z) = z$, $\delta_W(w) = (w, w^2, \ldots, w^{l-1})$

$$\mathbb{E}(Y \mid Z = \underset{\text{line in } z}{\underline{\underline{z}}}, W = w) = \underbrace{\beta_0 + \sum_{j=1}^{l-1} \beta_j^W w^j}_{\beta_0^Z(w) \text{ through polynomial in } w}$$

$$+ \underbrace{\left(\beta^Z + \sum_{j=1}^{l-1} \beta_j^{ZW} w^j\right)}_{\beta_1^Z(w) \text{ through polynomial in } w} \cdot z$$

In general:

- $\delta_W(w)$ determines on how $\mathbb{E}(Y \mid Z = z, W = w)$ depends on w for fixed (given) z (related coefficients then depend on z linearly through $\delta_Z(z)$)

- $\delta_Z(z)$ determines on how $\mathbb{E}(Y \mid Z = z, W = w)$ depends on z for fixed (given) w (related coefficients then depend on w linearly through $\delta_W(w)$)