5.4 Two categorical covariates

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NOW: two covariates Z, W

Z: categorical $\qquad$ W: categorical

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Example: $\quad$ = gender $\qquad$ = population

HowellsAll $\quad$ female / male $\qquad$ Australia / Berg / Burjati

$\in \{1, 2\}$ $\qquad$ $\in \{1, 2, 3\}$

$\in \{1, 2, .., G\}$ $\qquad$ $\in \{1, 2, .., H\}$

Z: parameterized by chosen (pseudo) contrasts

$$\mathbb{C} = \underbrace{\begin{pmatrix} c_1^T \\ \vdots \\ c_G^T \end{pmatrix}}_{(G-1)\text{ columns}} \qquad \text{i.e. } \delta_Z(z) = c_z$$

W: parameterized by chosen (pseudo-) contrasts

$$\mathbb{D} = \underbrace{\begin{pmatrix} d_1^T \\ \vdots \\ d_H^T \end{pmatrix}}_{(H-1)\text{ columns}} \qquad \text{i.e. } \delta_W(w) = d_w$$

5.4.1 Additivity

MODEL: $\mathbb{E}(Y \mid Z=z, W=w) = m(z,w) =: m_{z,w}$ (mean in subgroup labeled by $(Z=z, W=w)$)

$$= \beta_0 + \mathbf{c}_z^T \beta^Z + \mathbf{d}_w^T \beta^W$$

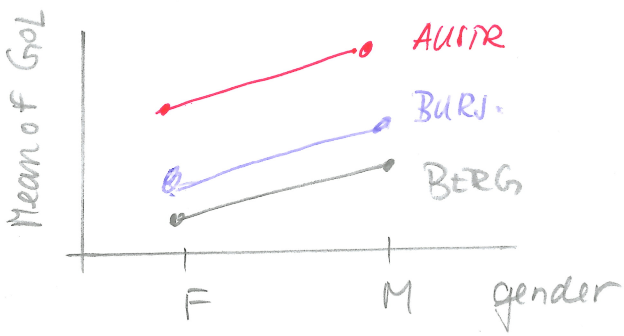
$$\beta^Z = (\beta_1^Z, \dots, \beta_{G-1}^Z)^T \qquad \beta^W = (\beta_1^W, \dots, \beta_{H-1}^W)$$

Additivity implies:

$$\mathbb{E}(Y \mid Z=j, W=w) - \mathbb{E}(Y \mid Z=\ell, W=w) = (\mathbf{c}_j^T - \mathbf{c}_\ell^T)\beta^Z \qquad \forall w = 1, \dots, H$$

$$\mathbb{E}(Y \mid Z=z, W=j) - \mathbb{E}(Y \mid Z=z, W=\ell) = (\mathbf{d}_j^T - \mathbf{d}_\ell^T)\beta^W \qquad \forall z = 1, \dots, G$$

GOL = glabell-occipital length (největší délka mozkovny)



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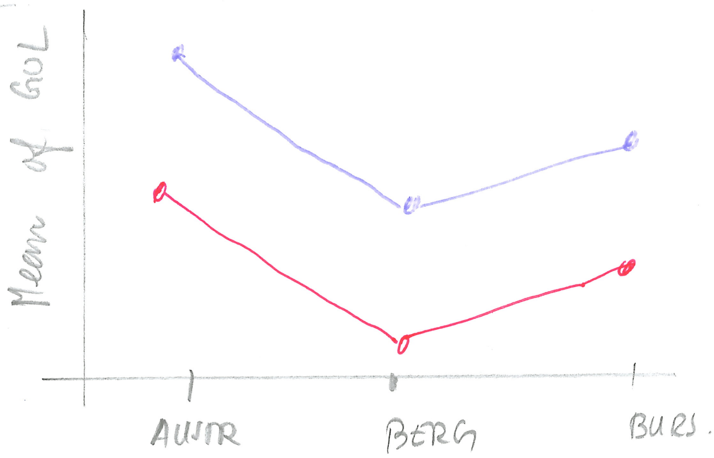
sample means per group on plots



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Reference group (pseudo) contrasts used for Z (and W)

$$\mathbb{C} = \begin{array}{c} \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ \vdots \\ G \end{array}$$

$$m_{g,h} := E(Y \mid Z=g, W=h) \underset{\text{ADDIT. MODEL}}{=} \beta_0 + c_g^T \underbrace{\beta^Z}_{=(\beta_1^Z, \dots, \beta_{G-1}^Z)^T} + d_h^T \beta^W$$

$g = 1, \dots, G, \quad h = 1, \dots, H$

$$\rightarrow \left.\begin{aligned} m_{1,h} &= \beta_0 + + d_h^T \beta^W \\ m_{2,h} &= \beta_0 + \beta_1^Z + d_h^T \beta^W \\ &\vdots \\ m_{G,h} &= \beta_0 + \beta_{G-1}^Z + d_h^T \beta^W \end{aligned}\right\} \text{ for arbitrary } h \in \{1, \dots, H\}$$

$$\rightarrow \beta_1^Z = m_{2,h} - m_{1,h} \left(= \underbrace{\frac{1}{H}\sum_{h=1}^{H} m_{2,h}} - \underbrace{\frac{1}{H}\sum_{h=1}^{H} m_{1,h}} \right)$$

$$= \overline{m}_{2\cdot} - \underbrace{\overline{m}_{1\cdot}}_{\text{reference}}$$

$$\vdots$$

$$\beta_{G-1}^Z = \underbrace{m_{G,h} - m_{1,h}}_{h \text{ arbitrary}} = \overline{m}_{G\cdot} - \overline{m}_{1\cdot}$$

(similar to Sec. 4.4)

- if also W parameterized by reference group (pseudo) contrasts $\mathbb{D} = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ 0 & \cdots & 1 \end{pmatrix}$

$\Rightarrow d_1^T \beta^W = 0$ and $\beta_0 = m_{1,1}$ (mean of group reference (reference))

That is $\beta^Z = (\beta_1^Z, \ldots, \beta_{G-1}^Z)^T$ provides

$$\beta_j^Z = \cancel{m_{j,h} - m} \quad m_{j+1,h} - m_{1,h} \quad (\text{for arbitrary } h)$$

$(j=1,\ldots,G-1)$

$$= \overline{m}_{j+1\bullet} - \overline{m}_{1\bullet}$$

$$= E(Y/Z=j+1, W=h) - E(Y/Z=1, W=h)$$

$$\beta_j^Z - \beta_l^Z = E(Y/Z=j+1, W=h) - E(Y/Z=l+1, W=h)$$

$$j,l = 1,\ldots,G-1$$

↳ set of (partial) effects of Z given $W=h$

$$= \overline{m}_{j+1\bullet} - \overline{m}_{l+1\bullet}$$

Analogously for W and $\beta^W = (\beta_1^W, \ldots, \beta_{H-1}^W)^T$:

$$\beta_j^W = m_{g,j+1} - m_{g,1} \quad (\text{for arbitrary } g)$$

$$= \overline{m}_{\bullet j+1} - \overline{m}_{\bullet 1}$$

$$= E(Y/Z=g, W=j+1) - E(Y/Z=g, W=1)$$

$$\beta_j^W - \beta_l^W = E(Y/Z=g, W=j+1) - E(Y/Z=g, W=l+1)$$

$$j,l = 1,\ldots,H-1$$

$$= \overline{m}_{\bullet j+1} - \overline{m}_{\bullet l+1}$$

R output: meaning of estimated β's

Sum contrasts used for Z (and W)

$$\mathbb{C} = \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \\ -1 & \cdots & -1 \end{pmatrix} \begin{array}{l} 1 \\ \\ G-1 \\ G \end{array}$$

$$m_{g,h} = \mathbb{E}(Y \mid Z=g, W=h)$$

$$\overset{\text{ADDIT. MODEL}}{=} \beta_0 + c_g^T \underbrace{\beta^Z}_{=(\beta_1^Z, \dots, \beta_{G-1}^Z)^T} + d_h^T \beta^W$$

$g = 1, \dots, G, \quad h = 1, \dots, H$

$$\left.\begin{aligned} \rightarrow\ m_{1,h} &= \beta_0 + \beta_1^Z + d_h^T \beta^W \\ &\ \vdots \\ m_{G-1,h} &= \beta_0 + \beta_{G-1}^Z + d_h^T \beta^W \\ m_{G,h} &= \beta_0 + \left(\sum_{j=1}^{G-1} (-\beta_j^Z)\right) + d_h^T \beta^W \end{aligned}\right\} \text{ for arbitrary } h = 1, \dots, H$$

$$\Rightarrow \frac{1}{G} \sum_{g=1}^{G} m_{g,h} \quad (=: \overline{m}_{\bullet h})$$

$$= \beta_0 + 0 + d_h^T \beta^W$$

$$\left.\begin{aligned} \Rightarrow\ \beta_1^Z &= m_{1,h} - \overline{m}_{\bullet h} \\ &\ \vdots \\ \beta_{G-1}^Z &= m_{G-1,h} - \overline{m}_{\bullet h} \end{aligned}\right\} \text{ for arbitrary } h = 1, \dots, H$$

$$\alpha_G^Z := -\sum_{j=1}^{G-1} \beta_j^Z = m_{G,h} - \overline{m}_{\bullet h}$$

If also W parameterized by sum contrasts

$$\Longrightarrow \beta_0 = \overline{m} := \frac{1}{G \cdot H} \sum_{g=1}^{G} \sum_{h=1}^{H} m_{g,h}$$

That is $\beta^Z = (\beta_1^Z, \dots, \beta_{G-1}^Z)^T$ provides

$$\beta_j^Z = m_{j,h} - \bar{m}_{\bullet h} \quad \text{(for arbitrary } h\text{)}$$
$$= \bar{m}_{j\bullet} - \bar{m}$$

$$\beta_j^Z - \beta_l^Z = m_{j,h} - m_{l,h} = E(Y/Z=j, W=h) - E(Y/Z=l, W=h)$$
$$j,l = 1, \dots, G-1$$
$$= \bar{m}_{j\bullet} - \bar{m}_{l\bullet}$$

$\Leftarrow$ again, set of (partial) effects of Z given $W=h$

$$\alpha_G^Z := -\sum_{j=1}^{G-1} \beta_j^Z = m_{G,h} - \bar{m}_{\bullet h} \quad \text{(for arbitrary } h\text{)}$$
$$= \bar{m}_{G\bullet} - \bar{m}$$

Analogously for W and $\beta^W = (\beta_1^W, \dots, \beta_{H-1}^W)^T$:

$$\beta_j^W = m_{g,j} - \bar{m}_{g\bullet} \quad \text{(for arbitrary } g\text{)}$$
$$= \bar{m}_{\bullet j} - \bar{m}$$

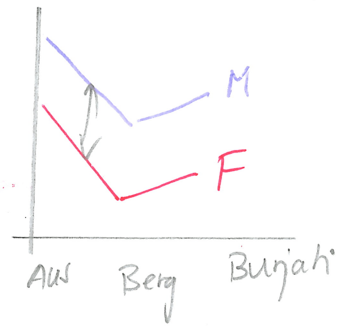
$$\beta_j^W - \beta_l^W = m_{g,j} - m_{g,l} = E(Y/Z=g, W=j) - E(Y/Z=g, W=l)$$
$$j,l = 1, \dots, H-1$$
$$= \bar{m}_{\bullet j} - \bar{m}_{\bullet l}$$

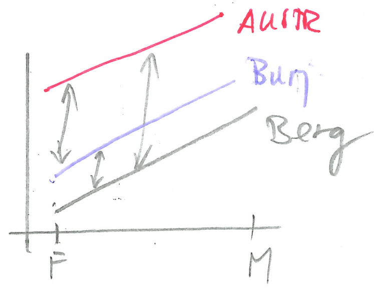
$$\alpha_H^W = -\sum_{j=1}^{H-1} \beta_j^W = m_{g,H} - \bar{m}_{g\bullet} \quad \text{(for arbitrary } g\text{)}$$
$$= \bar{m}_{\bullet H} - \bar{m}$$

5.4.2 Partial effects

Additivity: Effect of (change of) Z (or W) on the response expectation does not depend on a value of W (or Z).

MODEL: $\underbrace{\mathbb{E}(Y \mid Z=g, W=h)}_{m_{gh}} = \beta_0 + \mathbb{c}_g^T \beta^Z + \mathbb{d}_h^T \beta^W$





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Partial effects of a categorical covariate

Z: $\mathbb{E}(Y \mid Z=j, W=w) - \mathbb{E}(Y \mid Z=\ell, W=w) =$
$= (\mathbb{c}_j^T - \mathbb{c}_\ell^T)\beta^Z, \quad j, \ell = 1, \dots, G$

W: $\mathbb{E}(Y \mid Z=g, W=j) - \mathbb{E}(Y \mid Z=g, W=\ell) =$
$= (\mathbb{d}_j^T - \mathbb{d}_\ell^T)\beta^W, \quad j, \ell = 1, \dots, H$

No partial effect of Z given W
$\equiv H_0: \beta^Z = \mathbb{0}$

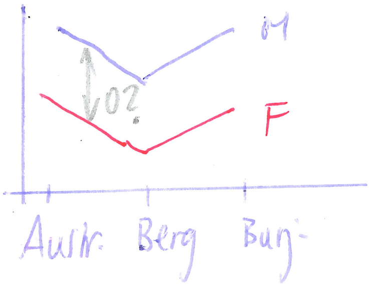
$\longrightarrow$ Wald F-test on a (subvector) of regression coefs.

$\longrightarrow$ Submodel F-test

$\beta^z = 0$ ≡ a (submodel) is obtained which assumes $E(Y \mid Z = g, W = h) =$

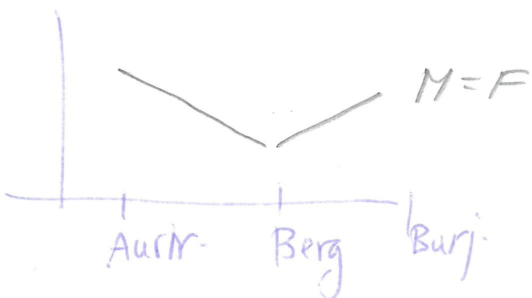
(bigger) model

$= \beta_0 + \varphi_h^T \beta^w$



→

submodel



R output

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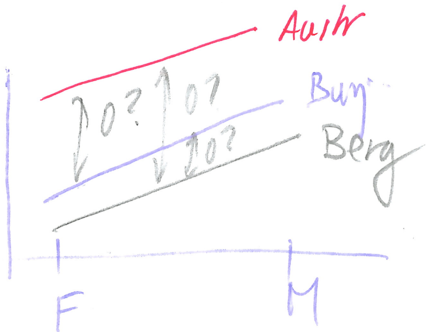
→ also called Two-way Analysis of Variance (Two-way ANOVA) (few more info later)



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drop1 in R

– both F-tests at once

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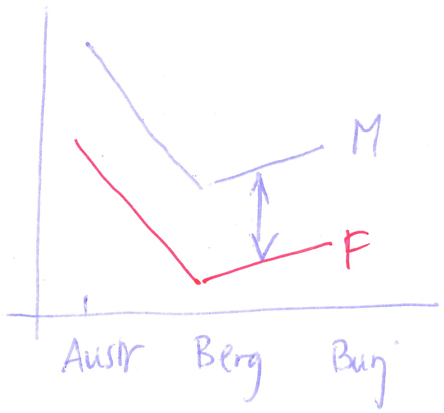
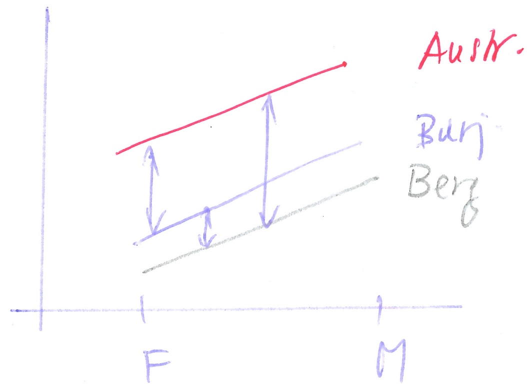
Quantification of both ~~parti~~ sets of partial effects

(1) Differences between conditional means





$E(Y \mid Z = \text{Male}, W = \cancel{h})$

$\quad - E(Y \mid Z = \text{Female}, W = \cancel{h})$

$= m_{2,\cancel{h}} - m_{1,\cancel{h}}$ (any h)

$(= \bar{m}_{2\bullet} - \bar{m}_{1\bullet})$

$E(Y \mid Z = g, W = \text{Berg})$

$\quad - E(Y \mid Z = g, W = \text{Austr.})$

etc.

$= m_{g,j} - m_{g,l}$ (any g)

$(= \bar{m}_{\bullet j} - \bar{m}_{\bullet l})$

→ most easily obtained from reference group (pseudo) contrast param. (contr. treatment)

Was derived:

$$\beta_j^Z = m_{j+1,h} - m_{1,h} \qquad j = 1, \dots, G-1$$

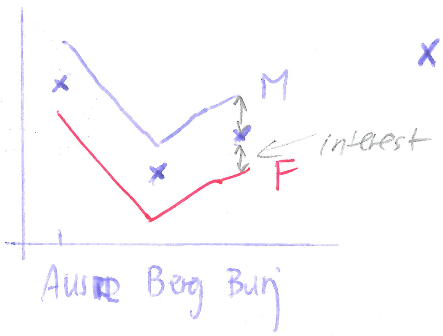
$$\beta_j^Z - \beta_l^Z = m_{j+1,h} - m_{l+1,h} \qquad j, l = 1, \dots, G-1$$

(analogously for W)

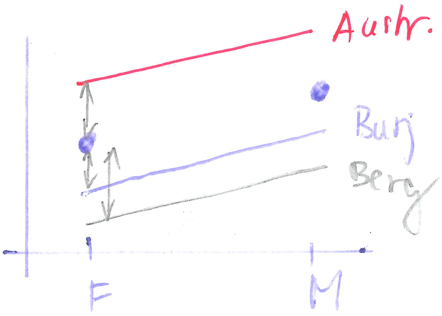
summary(…) 63

LSest(…) 64

(2) Differences between conditional means and means of the means by one factor



× $\overline{m}_{\bullet h} = \frac{1}{2}\left(E(Y/Z=M, W=h) + E(Y/Z=F, W=h)\right)$



• $\overline{m}_{g\bullet} = \frac{1}{3}\Big(E(Y/Z=g, W=\text{Austr.}) + E(Y/Z=g, W=\text{Berg}) + E(Y/Z=g, W=\text{Burj})\Big)$

Interest to know

$m_{j,h} - \overline{m}_{\bullet h} = \overline{m}_{j\bullet} - \overline{m}$

for arbitrary h

$j = 1,\ldots,G$

$m_{g,j} - \overline{m}_{g\bullet} = \overline{m}_{\bullet j} - \overline{m}$

for arbitrary g

$j = 1,\ldots,H$

→ most easily obtained from sum contrasts (contr. sum)

was derived:

$m_{j,h} - \overline{m}_{\bullet h}$

$= \overline{m}_{j\bullet} - \overline{m} \quad = \beta_j^Z, \quad j = 1,\ldots,G-1$

$m_{G,h} - \overline{m}_{\bullet h}$

$= \overline{m}_{G\bullet} - \overline{m} \quad = -\sum_{j=1}^{G-1} \beta_j^Z \quad (=: \alpha_G^Z)$

analogously for W:

$$\overline{m_{g,j}} - \overline{m_{g\cdot}} = \overline{m_{\cdot j}} - \overline{m} = \beta_j^W \quad , j = 1, .., H-1$$

$$\overline{m_{g,H}} - \overline{m_{g\cdot}} = \overline{m_{\cdot H}} - \overline{m} = -\sum_{j=1}^{H-1} \beta_j^W \quad (=: \alpha_H^W)$$

Remember order of levels

gender : 1= Female, 2= Male $\boxed{66}$

popul : 1= Australia, 2= Berg, 3= Burjah

LSest $\boxed{67}$

5.4.3 Interactions

- additivity does not have to hold

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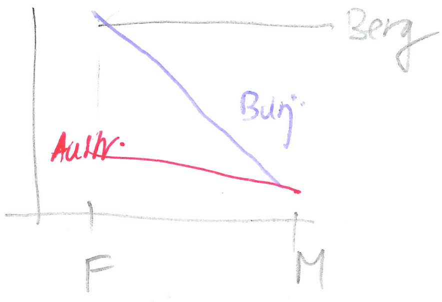

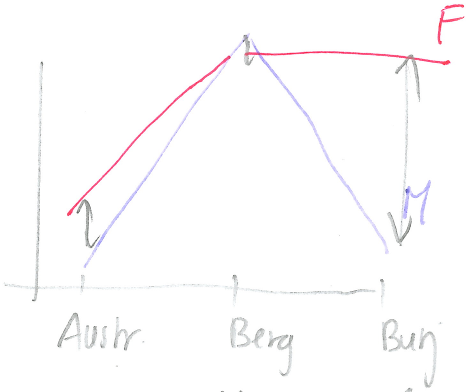

- differences between populations are different for male and female

- differences between M and F are different in diff populations

$\mathbb{E}(Y \mid Z=g, W=h) =: m_{g,h}$ ≡ set of general $G \cdot H$ group means

→ corresponding regression space must have a ~~rank~~ vector dimension $G \cdot H$

→ corresp. model matrix must have a rank $G \cdot H$

Interaction model

$$\underbrace{\mathbb{E}(Y \mid Z=g, W=h)}_{m_{g,h}} = \underbrace{\underset{1}{\beta_0} + \underset{G-1}{c_g^T \beta^Z} + \underset{H-1}{d_h^T \beta^W} + \underset{(G-1)\cdot(H-1)}{(d_h^T \otimes c_g^T)\beta^{ZW}}}_{\Sigma \,=\, G+H-1+GH-G-H+1 \,=\, G \cdot H}$$

With some effort, while using properties of Kronecker product, we can show that the rank of corresponding model matrix is $G \cdot H$ as soon as $n_{g,h} > 0 \ \forall g,h$, where $n_{g,h} = \#$obs. with $(Z=g, W=h)$

- different choices of (pseudo) contrasts lead to different interpretation of regression coefficients: $\beta_0, \beta^Z, \beta^W, \beta^{ZW}$

- to derive interpretation of β's, let

$$\alpha_0 = \beta_0, \quad \alpha_g^Z = c_g^T \beta^Z, \quad \alpha_h^W = d_h^T \beta^W,$$
$$\alpha_{g,h}^{ZW} = (d_h^T \otimes c_g^T)\,\beta^{ZW}$$

i.e. $$m_{g,h} = \beta_0 + c_g^T \beta^Z + d_h^T \beta^W + (d_h^T \otimes c_g^T)\,\beta^{ZW}$$
$$= \alpha_0 + \alpha_g^Z + \alpha_h^W + \alpha_{g,h}^{ZW}$$

$g = 1, \dots, G$
$h = 1, \dots, H$

Z \ W	1	$\dots$	H
1	$m_{1,1}$	$\dots$	$m_{1,H}$
$\vdots$	$\vdots$	$\dots$	$\vdots$
G	$m_{G,1}$	$\dots$	$m_{G,H}$

=

Z \ W	1	$\dots$	H	
1	$\alpha_{1,1}^{ZW}$	$\dots$	$\alpha_{1,H}^{ZW}$	$+\alpha_1^Z$
$\vdots$	$\vdots$			$\vdots$
G	$\alpha_{G,1}^{ZW}$	$\dots$	$\alpha_{G,H}^{ZW}$	$+\alpha_G^Z$
	$+\alpha_1^W$	$\dots$	$+\alpha_H^W$	$+\alpha_0$

$$\alpha_0 = \beta_0$$
$$\alpha_g^Z = c_g^T \beta^Z$$
$$\alpha_h^W = d_h^T \beta^W$$
$$\alpha_{g,h}^{ZW} = (d_h^T \otimes c_g^T)\,\beta^{ZW}$$

Reference group (pseudo) contrasts (contr. treatment) used for both Z and W

$$\mathbb{C} = \begin{array}{c} \\ \left(\begin{array}{ccc} 0 & \cdots & 0 \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{array} \right) \end{array} \begin{array}{c} 1 \\ 2 \\ \vdots \\ G \end{array} \Rightarrow \quad \begin{aligned} \alpha_1^Z &= \mathbb{C}_1^T \beta^Z = 0 \\ \alpha_2^Z &= \mathbb{C}_2^T \beta^Z = \beta_1^Z \\ &\vdots \\ \alpha_G^Z &= \mathbb{C}_G^T \beta^Z = \beta_{G-1}^Z \end{aligned}$$

$$\mathbb{D} = \left(\begin{array}{ccc} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ & \ddots & \\ 0 & & 1 \end{array} \right) \begin{array}{c} 1 \\ 2 \\ \vdots \\ H \end{array} \qquad \begin{aligned} \alpha_1^W &= \mathbb{d}_1^T \beta^W = 0 \\ \alpha_2^W &= \mathbb{d}_2^T \beta^W = \beta_1^W \\ &\vdots \\ \alpha_H^W &= \mathbb{d}_H^T \beta^W = \beta_{H-1}^W \end{aligned}$$

interaction terms: $\alpha_{g,h}^{ZW} = \underbrace{(\mathbb{d}_h^T \otimes \mathbb{C}_g^T)}\, \beta^{ZW}$

$= (0, \ldots, 0)^T$ if $h=1$ or $g=1$

$= (0, \ldots, 1, \ldots, 0)^T$ otherwise

$$\alpha_{g,h}^{ZW} = \begin{cases} 0, & g=1 \text{ or } h=1 \\ \beta_{g-1,h-1}^{ZW} & \text{otherwise} \end{cases}$$

$$\beta^{ZW} = (\beta_{1,1}^{ZW}, \ldots, \beta_{G-1,1}^{ZW}, \ldots, \beta_{1,H-1}^{ZW}, \ldots, \beta_{G-1,H-1}^{ZW})^T$$

$$m_{gh} = \alpha_0 + \alpha_g^Z + \alpha_h^W + \alpha_{g,h}^{ZW}$$

Z \ W	1	2	…	H
1	$m_{1,1}$	$m_{1,2}$	…	$m_{1,H}$
2	$m_{2,1}$	$m_{2,2}$	…	$m_{2,H}$
⋮				
G	$m_{G,1}$	$m_{G,2}$	…	$m_{G,H}$

Z \ W	1	2	…	H	
1	$\alpha_{1,1}^{ZW}$	$\alpha_{1,2}^{ZW}$	…	$\alpha_{1,H}^{ZW}$	$+\alpha_1^Z$
2	$\alpha_{2,1}^{ZW}$	$\alpha_{2,2}^{ZW}$	…	$\alpha_{2,H}^{ZW}$	$+\alpha_2^Z$
⋮					
G	$\alpha_{G,1}^{ZW}$	$\alpha_{G,2}^{ZW}$	…	$\alpha_{G,H}^{ZW}$	$+\alpha_G^Z$
	$+\alpha_1^W$	$+\alpha_2^W$	…	$+\alpha_H^W$	$+\alpha_0$

	W		
Z	1	2 …	H
1	$m_{1,1}$	$m_{1,2}$ …	$m_{1,H}$
2	$m_{2,1}$	$m_{2,2}$	$m_{2,H}$
⋮	⋮	⋮	⋮
G	$m_{G,1}$	$m_{G,2}$ …	$m_{G,H}$

$$m_{g,h} = \alpha_0 + \alpha_g^Z + \alpha_h^W + \alpha_{g,h}^{ZW}$$

	W				
Z	1	2 …	H		
1	0	0 …	0	$+0$	α_1^Z
2	0	$\beta_{1,1}^{ZW}$ …	$\beta_{1,H-1}^{ZW}$	β_1^Z	α_2^Z
⋮	⋮				
G	0	$\beta_{G-1,1}^{ZW}$	$\beta_{G-1,H-1}^{ZW}$	β_{G-1}^Z	α_G^Z
	$+0$	$+\beta_1^W$	$+\beta_{H-1}^W$	$+\beta_0$	
	α_1^W	α_2^W	α_H^W		

$\Longrightarrow$ $\alpha_0 = \beta_0 = m_{1,1}$ (= mean for Z = reference, W = reference)

$\alpha_g^Z = \beta_{g-1}^Z = m_{g,1} - m_{1,1}$, $g = 2, \ldots, G$

W = reference (Z, W) = (reference, reference)

$\alpha_h^W = \beta_{h-1}^W = m_{1,h} - m_{1,1}$

Z = reference

$\alpha_{g,h}^{ZW} = \beta_{g-1,h-1}^{ZW} = m_{g,h} - m_{g,1} - m_{1,h} + m_{1,1}$

R output $\boxed{71}$

Remember

gender: Female, Male

population: Australia, Berg, Burjah

reference

Sum contrasts (contr.sum)
used for both Z and W

$$\mathbb{C} = \begin{array}{c} \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \\ -1 & \cdots & -1 \end{pmatrix} \end{array} \begin{array}{l} 1 \\ \vdots \\ G-1 \\ G \end{array} \Rightarrow \quad \begin{aligned} \alpha_1^Z &= c_1^T \beta^Z = \beta_1^Z \\ &\ \vdots \\ \alpha_{G-1}^Z &= c_{G-1}^T \beta^Z = \beta_{G-1}^Z \\ \alpha_G^Z &= c_G^T \beta^Z = -\sum_{j=1}^{G-1} \beta_j^Z \end{aligned}$$

$$\mathbb{D} = \begin{array}{c} \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \\ -1 & \cdots & -1 \end{pmatrix} \end{array} \begin{array}{l} 1 \\ \vdots \\ H-1 \\ H \end{array} \Rightarrow \quad \begin{aligned} \alpha_1^W &= d_1^T \beta^W = \beta_1^W \\ &\ \vdots \\ \alpha_{H-1}^W &= d_{H-1}^T \beta^W = \beta_{H-1}^W \\ \alpha_H^W &= d_H^T \beta^W = -\sum_{j=1}^{H-1} \beta_j^W \end{aligned}$$

<u>interaction terms</u>: $\beta^{ZW} = (\underbrace{\beta_{1,1}^{ZW}, \ldots, \beta_{G-1,1}^{ZW}}_{\beta_{\bullet 1}^{ZW}}, \ldots, \underbrace{\beta_{1,H-1}^{ZW}, \ldots, \beta_{G-1,H-1}^{ZW}}_{\beta_{\bullet H-1}^{ZW}})^T$

<u>analogously</u>: $\beta_{g\bullet}^{ZW} := (\beta_{g,1}^{ZW}, \ldots, \beta_{g,H-1}^{ZW})^T$

$$\alpha_{g,h}^{ZW} = \underbrace{(d_h^T \otimes c_g^T)}_{= \begin{cases} (0, \ldots, 1, \ldots, 0)^T \\ (0 \ldots, 0, \underline{-1, \ldots, -1}, 0, \ldots 0)^T \\ (\underline{0, \ldots, -1, \ldots 0}, \ldots, \underline{0, \ldots, -1, \ldots 0})^T \\ (1, \ldots, 1)^T \end{cases}} \beta^{ZW} = \begin{cases} \beta_{g,h}^{ZW} \\ -\sum_{j=1}^{G-1} \beta_{j,h} \\ -\sum_{j=1}^{H-1} \beta_{g,j} \\ \sum_{j=1}^{G-1} \sum_{l=1}^{H-1} \beta_{j,l}^{ZW} \end{cases}$$

Z \ W	1	…	H-1	H
1	$m_{1,1}$	…	$m_{1,H-1}$	$m_{1,H}$
⋮				
G-1	$m_{G-1,1}$	…	$m_{G-1,H-1}$	$m_{G-1,H}$
G	$m_{G,1}$		$m_{G,H-1}$	$m_{G,H}$

Z \ W	1	…	H-1	H	
1	$\beta_{1,1}^{ZW}$	…	$\beta_{1,H-1}^{ZW}$	$-\sum_{j=1}^{H-1}\beta_{1,j}^{ZW}$	$+\beta_1^{Z}$
⋮	⋮				
G-1	$\beta_{G-1,1}^{ZW}$	…	$\beta_{G-1,H-1}^{ZW}$	$-\sum_{j=1}^{H-1}\beta_{G-1,j}^{ZW}$	$+\beta_{G-1}^{Z}$
G	$-\sum_{j=1}^{G-1}\beta_{j,1}^{ZW}$	…	$-\sum_{j=1}^{G-1}\beta_{j,H-1}^{ZW}$	$\sum_{j}^{G-1}\sum_{\ell}^{H-1}\beta_{j,\ell}^{ZW}$	$-\sum_{j=1}^{G-1}\beta_j^{Z}$
	$+\beta_1^{W}$	…	$+\beta_{H-1}^{W}$	$-\sum_{j=1}^{H-1}\beta_j^{W}$	$+\beta_0$

$$m_{g,h} = \alpha_0 + \alpha_g^{Z} + \alpha_h^{W} + \alpha_{g,h}^{ZW}$$

$\to$ α's satisfy:

$$\sum_{g=1}^{G}\alpha_g^{Z} = 0$$

$$\sum_{h=1}^{H}\alpha_h^{W} = 0$$

$$\sum_{h=1}^{H}\alpha_{g,h}^{ZW} = 0 \quad \forall g$$

$$\sum_{g=1}^{G}\alpha_{g,h}^{ZW} = 0 \quad \forall h$$

Interpretation of α's (β's are their subset):

$$m_{g,h} = \alpha_0 + \alpha_g^{Z} + \alpha_h^{W} + \alpha_{g,h}^{ZW},$$

NOW: $\sum_{g=1}^{G} \alpha_g^{Z} = 0$, $\sum_{h=1}^{H} \alpha_h^{W} = 0$

$\sum_{g=1}^{G} \alpha_{g,h}^{ZW} = 0 \quad \forall h$

$\sum_{h=1}^{H} \alpha_{g,h}^{ZW} = 0 \quad \forall g$

$$\Rightarrow \underbrace{\frac{1}{G \cdot H} \sum_{g=1}^{G} \sum_{h=1}^{H} m_{g,h}}_{=: \overline{m}} = \alpha_0 \; (+0+0+0)$$

$$\alpha_0 = \overline{m}$$

$$\underbrace{\frac{1}{G} \sum_{g=1}^{G} m_{g,h}}_{\overline{m}_{\bullet h}} = \alpha_0 + (0) + \alpha_h^{W} \; (+0)$$

$$\Rightarrow \alpha_h^{W} = \overline{m}_{\bullet h} - \overline{m}$$

$$\frac{1}{H} \sum_{h=1}^{H} m_{g,h} = \alpha_0 + \alpha_g^{Z} + (0+0)$$

$$\Rightarrow \alpha_g^{Z} = \overline{m}_{g\bullet} - \overline{m}$$

$$\alpha_{g,h}^{ZW} = m_{g,h} - \overline{m}_{g\bullet} - \overline{m}_{\bullet h} + \overline{m}$$

R output $\boxed{72}$

$Z \in \{$Female, Male$\}$

$W \in \{$Austr., Berg, Bunjah$\}$

NOTE: In particular quantities

$$\alpha_g^{Z} = \overline{m}_{g\bullet} - \overline{m}$$

and $\alpha_h^{W} = \overline{m}_{\bullet h} - \overline{m}$

are of interest in case of designed (industrial) experiments.

$Z \in \{1, \dots, G\}$: G settings of a concentration

$W \in \{1, \dots, H\}$: H settings of temperature

5.4.4 Additivity or interactions ?

Can additivity be assumed?

General parametrisation of

$$m_{g,h} = \mathbb{E}(Y \mid Z = g, W = h) =$$

$$= \beta_0 + \mathbf{c}_g^T \beta^Z + \mathbf{d}_h^T \beta^W + (\mathbf{d}_h^T \otimes \mathbf{c}_g^T)\beta^{ZW}$$

$(G-1)\cdot(H-1)$ params.

if $n_{g,h} > 0 \ \forall (g,h)$

→ related linear model is of full-rank

ADDITIVITY ≡ $H_0: \beta^{ZW} = 0$

→ Wald test on a subset of regression coefficients

→ submodel F-test with submodel = additivity model

Plot gol 74

R: submodel F-test 75

Plot oca 76

R: submodel F-test 77