Minions of nilpotent groups

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AAA109

Ljubljana, 5.6.2026

Minions

Def. An (abstract) minion $\mathcal{M}$ is given by

$$\mathcal{M}(X) \qquad \forall \text{ finite set } X$$

$$\mathcal{M}(\alpha) : \mathcal{M}(X) \longrightarrow \mathcal{M}(Y) \quad \forall \alpha : X \longrightarrow Y$$

s.t. $\mathcal{M}(\alpha \circ \beta) = \mathcal{M}(\alpha) \circ \mathcal{M}(\beta)$

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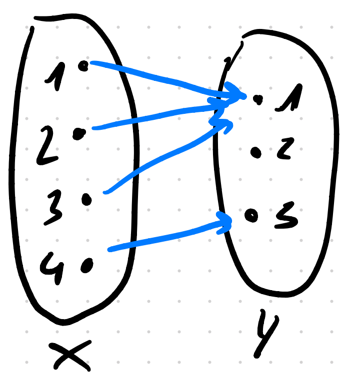
<u>Example</u>

$$\mathcal{M}(X) = \mathbb{Z}^X$$

α :



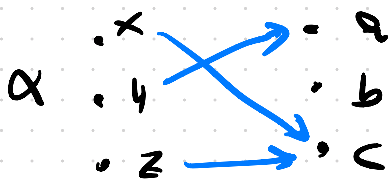
$$\mathcal{M}(\alpha): \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}}_{\in \mathbb{Z}^X} \mapsto \underbrace{\begin{pmatrix} a_1 + a_2 + a_3 \\ 0 \\ a_4 \end{pmatrix}}_{\in \mathbb{Z}^Y}$$

Group/Semigroup minions

Example

-) $\mathcal{M}(X) = X^*$ (free monoid over X)

$\mathcal{M}(\alpha) : X^* \longrightarrow Y^*$ substitute letters via α



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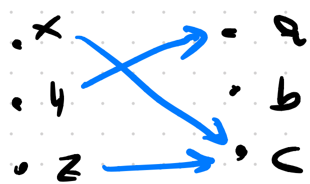
α 

$\mathcal{M}(\alpha)(yxzyy) = accaa$

-) $\mathcal{M}(X) = F(X)$ (free group over X)

$$\begin{tikzcd} X \arrow[r, "\alpha"'] \arrow[d, hook] & Y \arrow[d, hook] \\ F(X) \arrow[r, "\mathcal{M}(\alpha)"'] & F(Y) \end{tikzcd}$$

homomorphism extending α

Minions from varieties

For every variety $\mathcal{V}$ $\exists$ minion $\mathcal{M}_{\mathcal{V}}$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, hook] & Y \arrow[d, hook] \\ F_{\mathcal{V}}(X) \arrow[r, "\mathcal{M}(\varphi)_{\mathcal{V}}"'] & F_{\mathcal{V}}(Y) \end{tikzcd}$$

Task: find nice description of $\mathcal{M}_{\mathcal{V}}$ for

$\mathcal{V}$ = groups

= n-nilpotent groups

= $HSP(\underline{G})$, $\underline{G}$ finite group $\quad (\mathcal{M}_{\mathcal{V}} = \mathrm{Clo}(\underline{G}))$

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multilinear?

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Subword count

for $w \in X^*$ define $f_w : X^* \to \mathbb{N}$

$$f_w(v) = \#\text{ subwords } = v \text{ in } w$$

<u>Example</u> $X = \{a, b, c\}$

$$w = bbaacbb$$

v	a	bc	ca	bcb
$f_w(v)$	2	2	0	4

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Identities by counting pairs of subwords

$$(*)\ \begin{cases} f(x) \cdot f(y) = f(xy) + f(yx) & \forall x \neq y \in X \\ f(x)^2 = 2f(xx) + f(x) & \\ f(xy) \cdot f(z) = f(xyz) + f(xzy) + f(zxy) & z \neq x, y \\ \vdots & \end{cases}$$

Subword count minion

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Def. $M^+(X) := \{ f : X^* \to \mathbb{N} \mid f \text{ satisfies } (*) \}$

$$(f\cdot g)(w) := \sum_{w=w_1w_2} f(w_1)\cdot g(w_2)$$

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Theorem [MK'26] X finite $\Rightarrow X^* \cong M^+(X)$

M^+ is a minion, with minors $f^\alpha = M^+(\alpha)(f)$

$$\boxed{f^\alpha(y_1y_2\cdots y_k) = \sum_{\alpha(x_i)=y_i} f(x_1x_2\cdots x_k)}$$

„Subword count" for groups

Identities by counting pairs of subwords

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For finite X

$$\begin{tikzcd} X^* \arrow[r, "\cong"] \arrow[d, hook] & M^+(X) = \{ f : X^* \to \mathbb{N} \mid f \text{ satisfies } \circledast \} \arrow[d, hook] \\ F(X) \arrow[r, hook, "\iota"] & M(X) := \{ f : X^* \to \mathbb{Z} \mid f \text{ satisfies } \circledast \} \end{tikzcd}$$

ι is embedding

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$$\begin{array}{ccl} X^* & \cong & M^+(X) = \{ f : X^* \to \mathbb{N} \mid f \text{ satisfies } (*) \} \\ \downarrow & & \downarrow \\ F(X) & \overset{\iota}{\hookrightarrow} & M(X) := \{ f : X^* \to \mathbb{Z} \mid f \text{ satisfies } (*) \} \end{array}$$

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⚠ for $w \in F(X) \setminus X^*$ no „subword count" interpretation

e.g. $w = x^{-2}$

v	x	x^2	x^3	x^4	x^5
$c(w)(v)$	-2	3	-4	5	-6

"Subword count" for groups

$F(X) \overset{\iota}{\hookrightarrow} M(X) := \{ f : X^* \to \mathbb{Z} \mid f \text{ satisfies } \circledast \}$

Let $G_1 = F(X)$, $G_{n+1} = [G_n, G_1]$

Lemma $w \in G_n \iff \iota(w)(v) = 0 \quad \forall |v| < n$

Example

$w = [x, y] = x^{-1} y^{-1} x y$

v	x	y	x^2	y^2	xy	yx
$\iota(w)(v)$	0	0	0	0	1	-1

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Proof idea

induction over commutator terms

A minion for nilpotent groups

$F_n(X) := F(X)/G_{n+1}$ … free n-step nilpotent group

$\iota : F_n(X) \hookrightarrow M_n(X) := \{ f : X^{\leq n} \longrightarrow \mathbb{Z} \mid f \text{ satisfies } (*) \}$

(elementary check) ι is isomorphism for $n = 1, 2, 3, 4$

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WORK IN PROGRESS

$M_n(X) \overset{?}{\cong} F_n(X) \quad \forall n \in \mathbb{N}$

$M(X) = \overline{\iota(F(X))}$ (ptw. convergence)

If true [Barto, Zhuk, M.K. ...] $\overset{?}{\rightsquigarrow}$ „n-th level" of AIP relaxations solves n-nilpotent group CSPs

What about finite nilpotent groups?

$$M_n^q(X) := \{ f : X^{\leq n} \to \mathbb{Z}_q \mid f \text{ satisfies } \circledast \}$$

Theorem [Thérien '83, MK '26]

1) $\forall\, \underline{G}$ finite, n-nilpotent $q = \exp(\underline{G})$:

$$\exists\, \xi : M_n^q \to \mathrm{Clo}(\underline{G}) \text{ minion homom.}$$

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$\exists\, \xi : M_n^q \to \mathrm{Clo}(\underline{G})$ minion homom.

2) $\forall \underline{G}$ finite p-group $\exists n$: ⚠ possibly $n >$ nilpotence degree

$\exists\, \xi : M_n^p \to \mathrm{Clo}(\underline{G})$ minion homom.

Examples

Minions — **CSP** [Barto, Hadek, Zhuk '26]

$\mathrm{Clo}(\underline{D_8}), \mathrm{Clo}(\underline{Q_8}) \cong \mathcal{M}_2^2$

solved by '2nd level' of $\mathbb{Z}_2$

$\mathcal{M}^{p}_{p^{n-1}} \to \mathrm{Clo}(\mathbb{Z}_{p^n})$

(by Lucas's theorem)

$\mathrm{CSP}(\mathbb{Z}_{p^2})$ solved by 'p-th level' of $\mathbb{Z}_p$

Thank you!

Questions?
Answers?
Counterexamples?