Dichotomy results for constraint satisfaction problems

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Constraint satisfaction problems (CSPs)

Let $\mathcal{A}$ be a structure in a finite language L

CSP($\mathcal{A}$)

Instance: $\psi = \exists x_1, \ldots, x_j \, \phi_1 \wedge \ldots \wedge \phi_n$ with ϕ_i atomic L-formulas
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The input is called a primitively positive sentence (pp-sentence).

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$$R_1(x,y) :\Leftrightarrow x \vee y$$
$$R_2(x,y) :\Leftrightarrow x \vee \neg y$$
$$R_3(x,y) :\Leftrightarrow \neg x \vee \neg y$$

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If R is pp-definable in Ψ then $\mathrm{CSP}(\{0,1\},R,\Psi)$ reduces to $\mathrm{CSP}(\{0,1\},\Psi)$.

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Classify templates up to primitive positive interdefinability.

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Schaefer '79

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Proven for $|\mathcal{A}| \leq 3$ (Bulatov '06).

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- random graph (V, E): finite graphs

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Betweenness

Instance: Given a set of variables and triples (x, y, z)
Problem: Is there a linear order on the variables such that $\mathrm{Betw}(x, y, z)$ for all triples?
$\mathrm{CSP}(\mathbb{Q}, \mathrm{Betw})$

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Classify all the reducts of $(\mathbb{Q}, <)$, up to pp-interdefinability

Polymorphism clones

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$$(x, y) \mapsto \min(x, y) \in \mathrm{Pol}(\mathbb{Q}, <) \text{ since}$$
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$$\min \notin \mathrm{Pol}(\mathbb{Q}, \mathrm{Betw}) \text{ since}$$
$$\mathrm{Betw}(-1, 0, 1), \mathrm{Betw}(2, 0, -1), \neg\,\mathrm{Betw}(-1, 0, -1)$$

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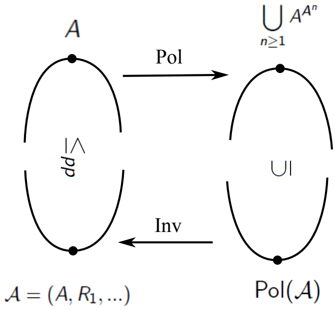

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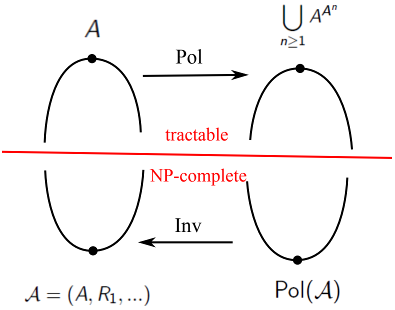
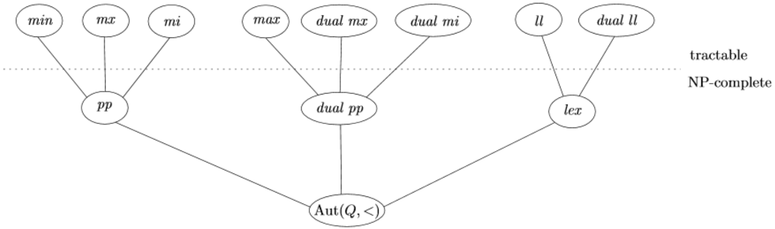


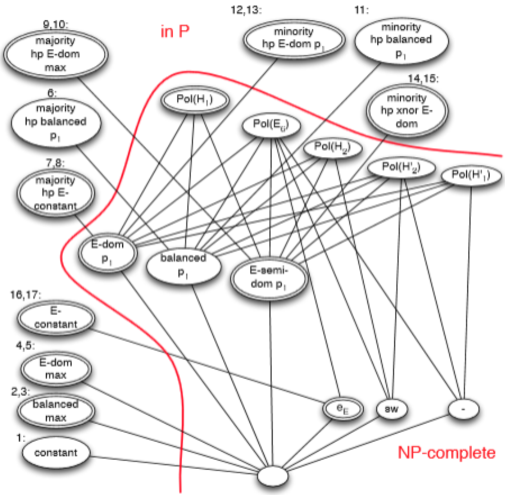
Dichotomy for Temp-SAT

Theorem (Bodirsky + Kára, '10)

Let $\mathbb{Q}_\Psi$ be a reduct of $(\mathbb{Q}, <)$. If $\mathrm{Pol}(\mathbb{Q}_\Psi)$ contains one of the operators *ll, min, mi, mx, their duals, or a constant operation*, then it is tractable. Otherwise it lies in NP-complete.



Dichotomy for Graph-SAT



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- Compare the complexity of CSPs on different domains
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Thank you!