A complexity dichotomy for poset constraint satisfaction

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Outline

1. **Poset-SAT**
2. Poset-SAT as CSP over the random partial order
3. The universal algebraic approach
4. Results

Boolean-SAT

Let Φ be a finite set of propositional formulas.

Boolean-SAT(Φ)

Instance:

- Variables $\{x_1, \ldots, x_n\}$ and
- finitely many formulas $\phi_i(x_{i_1}, \ldots, x_{i_k})$, where each $\phi_i \in \Phi$.

Question:

Is $\bigwedge \phi_i(x_{i_1}, \ldots, x_{i_k})$ satisfiable in $\{0, 1\}$?

Computational complexity is in NP and depends on Φ.

Theorem (Schaefer '78)

For every Φ, Boolean-SAT(Φ) is either in P or in NP-complete.

Poset-SAT

Let Φ be a finite set of *quantifier-free* $\{\leq\}$*-formulas*

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Question:

For which Φ is Poset-SAT(Φ) in P? For which NP-complete?

Examples

Poset-SAT($<$)

Instance: Variables $\{x_1, \ldots, x_n\}$ and formulas $x_{i_1} < x_{i_2}$.
Question: Is $\bigwedge(x_{i_1} < x_{i_2})$ satisfiable in a partial order?

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Poset-SAT($\perp$,Q)

$x \perp y := \neg(x \leq y) \wedge \neg(y \leq x)$
$Q(x, y, z) := (x < y \vee x < z)$

Poset-SAT($\perp$, Q) is NP-complete.

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Problem: How to determine the complexity for *every* Φ?

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An instance $\bigwedge \phi_i(x_{i_1}, \ldots, x_{i_k})$ of Poset-SAT(Φ) has a solution iff

$$(P; R_\phi)_{\phi \in \Phi} \models \exists x_1, \ldots, x_n \bigwedge R_{\phi_i}(x_{i_1}, \ldots, x_{i_k}).$$

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We call $(P; R_\phi)_{\phi \in \Phi}$ a reduct of $\mathbb{P}$.

CSPs over the random partial order

Let Γ be a reduct of $\mathbb{P}$.

CSP(Γ)

Instance: pp-formula $\exists x_1, \ldots, x_n \bigwedge R_{\phi_i}(x_{i_1}, \ldots, x_{i_k})$
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Easy observation

$\Gamma \leq_{pp} \Delta \rightarrow \mathrm{CSP}(\Gamma) \leq_p \mathrm{CSP}(\Delta)$.

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Let $\mathrm{Pol}(\Gamma)$ be the polymorphism clone of Γ, i.e. for an $f : P^n \to P$, $f \in \mathrm{Pol}(\Gamma)$ if for all relations R of Γ:

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For ω-categorical structure Γ, Δ we have

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$\to$ Aim: Understand polymorphism clones of reducts of $\mathbb{P}$!

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Preclassification by unary functions

Theorem (Pach, Pinsker, Pongrácz, Szabó '14)

Let Γ be a reduct of $\mathbb{P}$. Then $\mathrm{Aut}(\Gamma)$ is equal to one of the following:

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{Sym}(P) \arrow[d, no head] & \\
 & \langle \updownarrow, \circlearrowright \rangle \arrow[dl, no head] \arrow[dr, no head] & \\
\langle \updownarrow \rangle \arrow[dr, no head] & & \langle \circlearrowright \rangle \arrow[dl, no head] \\
 & \mathrm{Aut}(\mathbb{P}) &
\end{tikzcd}
$$

$\updownarrow$: bijection with $x < y \leftrightarrow \updownarrow x > \updownarrow y$

$\circlearrowright$: "rotation" at a generic upwards-closed set

Preclassification by unary functions

Proposition (K., Pham '16)

Let Γ be reduct of $\mathbb{P}$. Then the unary part of $\mathrm{Pol}(\Gamma)$ contains

1. a constant
2. or $g_<$ that maps P to a chain $\cong \mathbb{Q}$,
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$\rightarrow$ We only need to study Case 4

Polymorphisms of higher arity

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Bodirsky, Chen, Kára, von Oertzen '09

If $e_\leq \in \mathrm{Pol}(\Gamma)$ every relation in Γ is $\leq$-Horn:

$$(x_{i_1} \leq x_{j_1}) \wedge \cdots \wedge (x_{i_n} \leq x_{j_n}) \to (x_{i_{n+1}} \leq x_{j_{n+1}}) \text{ or}$$
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$\to$Use *Ramsey theory* and the method of *canonical functions*.

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Method by Bodirsky & Pinsker (very roughly):

If R not pp-definable in Γ there is an $f \in \mathrm{Pol}(\Gamma)$ violating R.
By Ramsey property there is also $g \in \mathrm{Pol}(\Gamma)$ violating R that is canonical $(P; \leq, \prec, \bar{c})^n \to (P; \leq)$.

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$\to$ Look for relations that imply NP-hardness.
$\to$ Use canonical functions for P.

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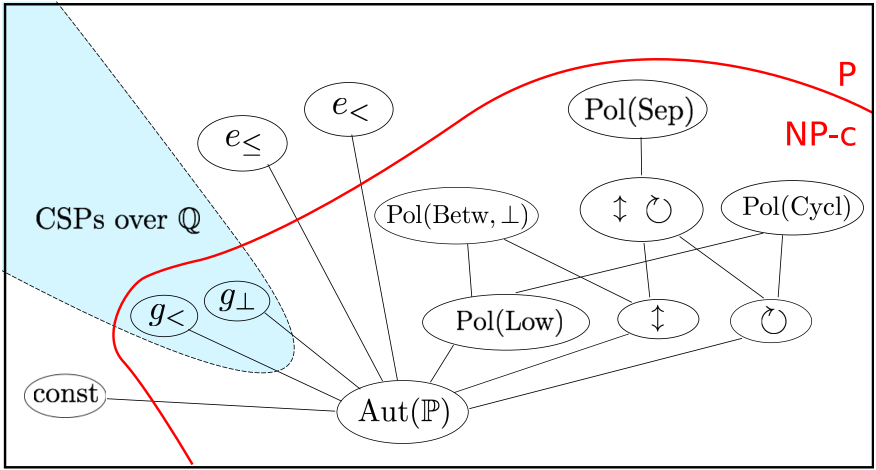
Lattice of polymorphism clones



Complexity dichotomy

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Consequence:

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Consequence:

Poset-SAT(Φ) is in P or NP-complete.
Given Φ, it is decidable to tell if Poset-SAT(Φ) is in P.

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- or Γ is homomorphic equivalent to a Δ, such that:

$$\xi : \mathrm{Pol}(\Delta, c_1, \ldots, c_n) \to \mathcal{P}$$

  and $\mathrm{CSP}(\Gamma)$ is NP-complete.

Thank you!