Completing labelled graphs to metric spaces

Michael Kompatscher

May 30, 2017

Joint work with
Andrés Aranda, David Bradley-Williams, Jan Hubička, Miltiadis Karamanlis, Matěj Konečný, Micheal Pawliuk

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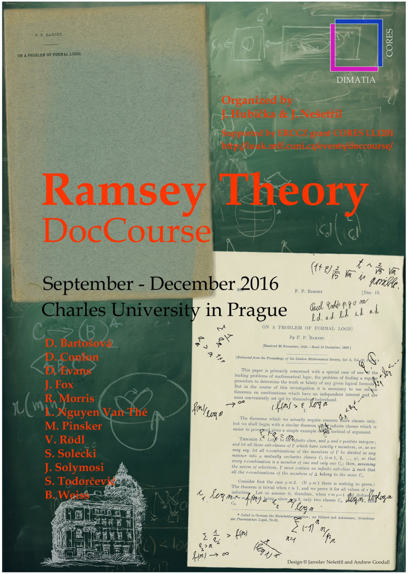
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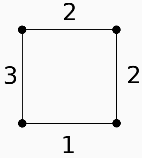
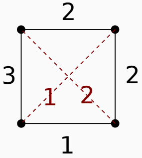
Introduction

Edge-labelled graphs

Every metric space can be regarded as an edge-labelled, complete graph:



(G, d)



a completion $(G, \bar{d})$

Questions

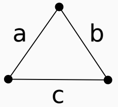
Given an edge-labelled graph (G, d):

- Can (G, d) be completed to a metric space $(G, \bar{d})$?
- Is there an algorithm completing (G, d)?
- Are there completion algorithms that preserves nice properties of the graph?

We consider metric spaces with distances $1, 2, \ldots, \delta$.

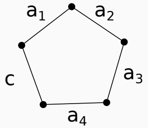
Non-metric cycles

By the triangle inequality, $(G, \bar{d})$ is not a metric space, if it contains a non-metric triangle.



$$a+b<c$$

More general, $(G, \bar{d})$ is not a metric space, if it contains a non-metric cycle.

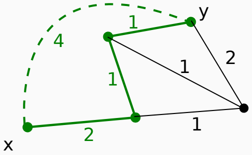


$$a_1 + a_2 + \dots + a_n < c$$

Non-metric cycles are obstacles, i.e. as subgraphs of (G, d) they prevent completion. In our setting there are only finite non-metric cycles.

Path completion

For a given edge-labelled graph (G, d), and non-edge (x, y), let $d^+(x, y)$ be the minimal path length between x any y:



$d^+(x, y) := \min(\delta, \min\{\sum_{i=0}^{k} d(u_i, u_{i+1}) : u_0 = x, u_{k+1} = y\})$

For all existing edges: $d^+(x, y) = d(x, y)$.

Let us call (G, d^+) the path completion of (G, d).

Path completion is correct

Lemma

The following are equivalent:

- (G, d^+) is a metric space
- (G, d) can be completed to a metric space
- (G, d) contains no non-metric cycles

Proof.

Assume that there is a non-metric triangle in (G, d^+), i.e. $d^+(u, v) + d^+(v, w) < d^+(u, w)$.

Then $d^+(u, w) = d(u, w)$ and there was already a non-metric cycle in (G, d). $\square$

Path completion is optimal

Lemma

The path completion maximizes distances: For (G, d), let $(G, \bar{d})$ be any completion to a metric space. Then

$$\bar{d}(x, y) \leq d^{+}(x, y) \leq \delta$$

Proof.

Assume $\bar{d}(x, y) > d^{+}(x, y)$. Then, $\bar{d}(x, y)$ and the path witnessing $d^{+}(x, y)$ form a non-metric cycle in $(G, \bar{d})$. $\square$

Path completion preserves automorphisms

Lemma

For all edge-labelled (G, d) we have $\mathrm{Aut}(G, d) \leq \mathrm{Aut}(G, d^+)$

Proof.

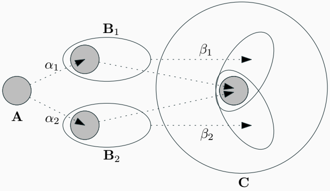
Let $f \in \mathrm{Aut}(G, d)$.
Note that $u_0, u_1, \ldots, u_l$ is a path from x to y, if and only if $f(u_0), f(u_1), \ldots, f(u_l)$ is a path from $f(x)$ to $f(y)$.
So $d^+(f(x), f(y)) = d^+(x, y)$, thus $f \in \mathrm{Aut}(G, d^+)$. $\square$

In general, completions do not have to preserve automorphisms.

Path completion implies amalgamation

Definition

We say that a class $\mathcal{C}$ of structures has the amalgamation property if



$\forall A, B_1, B_2 \in \mathcal{C}$, $\forall \alpha_i : A \to B_i$ $\exists C \in \mathcal{C}, \beta_i : B_i \to C$: $\beta_2\alpha_2 = \beta_1\alpha_1$.

The class of metric spaces (distances $1, 2, \ldots, \delta$) has a canonical amalgamation:

Take $C = B_1 \cup B_2$ and form its path completions.

Summary

Summary

Let (G, d^+) be the path completion of (G, d):

- (G, d^+) is metric if and only if (G, d) does not contain non-metric cycles
- $\mathrm{Aut}(G, d) \leq \mathrm{Aut}(G, d^+)$
- $d^+(x, y)$ is the maximal possible distance between x and y
- gives us a canonical amalgamation on metric spaces

Cherlin's census of metrically homogeneous graphs

Cherlin's census of metrically homogeneous graphs

Figure 1: Gregory Cherlin, likes to classify things

Cherlin's census of metrically homogeneous graphs

Figure 1: Gregory Cherlin, likes to classify things

In ongoing work, Cherlin gives a (probably) complete list of amalgamation classes of metric spaces that contain all geodesics, i.e. triangles $(a, b, |b - a|)$.

Cherlin's census of metrically homogeneous graphs

Cherlin '16

For parameters $(\delta, K_1, K_2, C_0, C_1)$ let $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ be the class of metric spaces of diameter δ such that for all triangles abc with $p = a + b + c$:

- $p < C_0$ if p is even
- $p < C_1$ if p is odd
- $2K_1 < p < 2K_2 + \min(a, b, c)$ if p is odd

Then $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ is an amalgamation class if and only if [see T-shirt].

Question

Is there an algorithm that completes edge-labelled graphs to $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$?

Cherlin's census of metrically homogeneous graphs

Cherlin light

For parameters (δ, K, C) let $\mathcal{A}^{\delta}_{K,C}$ be the class of metric spaces of diameter δ such that for all triangles abc with $p = a + b + c$:

- $p < C$
- $2K < p$ if p is odd

If $C > 2\delta + K$, then $\mathcal{A}^{\delta}_{K,C}$ is an amalgamation class.

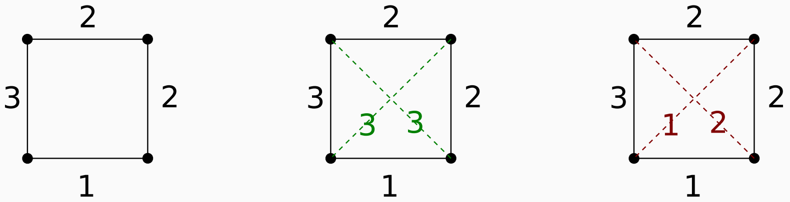
Question

Is there an algorithm that completes edge-labelled graphs to $\mathcal{A}^{\delta}_{K,C}$?

Path completion fails for $\mathcal{A}^{\delta}_{K,C}$

Adding big distances might introduce triangles of perimeter $> C$

Example: $\delta = 3$, $K = 1$, $C = 8$



Path completion (green) is not in $\mathcal{A}^{3}_{1,8}$, while the red completion is!

$\rightarrow$ Idea: optimize not towards δ, but to some $M < \delta$.

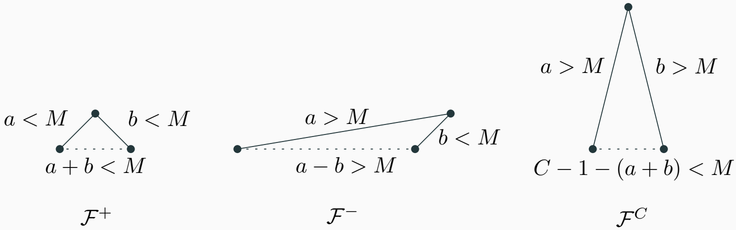
Only makes sense for $M \geq \frac{\delta}{2}$, $M \geq K$ and $M \leq \frac{C-\delta-1}{2}$.

Completing triangles

Optimizing distances towards $\max(K, \frac{\delta}{2}) \leq M \leq \frac{C-\delta-1}{2}$.

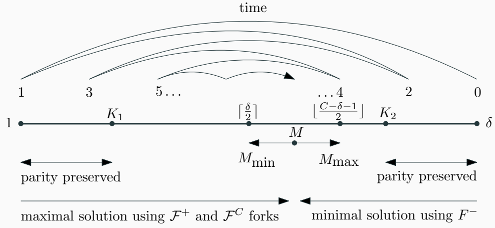
Triangles *MMa* are not forbidden due to the choice of M.

How to complete *forks*, i.e. triangles missing one edge?



Generalized M-completion of (G, d)

Add all new edges of length $t(i)$ to (G, d) in step i.




```
for i = 0 ...  delta-1 {
   if t(i) > M then complete all forks ab with b-a  = t(i)
   if t(i) < M then
      complete all forks ab with b+a  = t(i)
      complete all forks ab with C-b-a-1  = t(i)
        }
label remaining pairs by M
```

Properties of the completion algorithm

Optimization lemma

Let (G, d) be an edge-labelled graph, let $(G, \bar{d}) \in \mathcal{A}^{\delta}_{K,C}$ be a completion and let (G, d^M) be its M-completion. Then, for all $x, y \in G$:

$$\bar{d}(x, y) \geq d^M(x, y) \geq M \text{ or } \bar{d}(x, y) \leq d^M(x, y) \leq M.$$

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Proposition

Let (G, d) be an edge-labelled graph, let $(G, \bar{d}) \in \mathcal{A}^{\delta}_{K,C}$ be a completion and let (G, d^M) be its M-completion.

- $(G, d^M) \in \mathcal{A}^{\delta}_{K,C}$ and
- $\mathrm{Aut}(G, d) \leq \mathrm{Aut}(G, d^M)$.

Properties of the completion algorithm

Optimization lemma

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Proposition

Let (G, d) be an edge-labelled graph, let $(G, \bar{d}) \in \mathcal{A}^{\delta}_{K,C}$ be a completion and let (G, d^M) be its M-completion.

- $(G, d^M) \in \mathcal{A}^{\delta}_{K,C}$ and
- $\mathrm{Aut}(G, d) \leq \mathrm{Aut}(G, d^M)$.

$\rightarrow$ there is a finite set $\mathcal{O}$ of cycles that are obstacles to the completion.

Summary

Summary

Let (G, d^M) be the M-completion of (G, d):

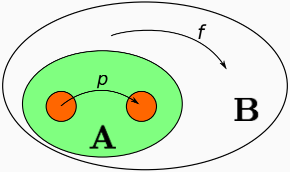
- $(G, d^M) \in \mathcal{A}^{\delta}_{K,C}$
  $\Leftrightarrow (G, d)$ has a completion in $\mathcal{A}^{\delta}_{K,C}$
  $\Leftrightarrow (G, d)$ does not contain a cycle $\in \mathcal{O}$
- $\mathrm{Aut}(G, d) \leq \mathrm{Aut}(G, d^M)$
- The distance $d^M(x, y)$ is the closest possible to M
- M-completion gives us a canonical amalgamation on $\mathcal{A}^{\delta}_{K,C}$

The extension property for partial automorphisms (EPPA)

The extension property for partial automorphisms (EPPA)

Question

Let $\mathcal{C}$ be a class of finite structures. Given a $\mathbf{A} \in \mathcal{C}$ and a set I of partial automorphisms of $\mathbf{A}$. Is there a structure $\mathbf{A} \leq \mathbf{B} \in \mathcal{C}$ such that $p \in I$ extends to an automorphism $f \in \mathrm{Aut}(\mathbf{B})$?



We say $\mathcal{C}$ has the extension property for partial automorphisms (EPPA) (or Hrushovski property) if the above is true for all $\mathbf{A} \in \mathcal{C}$.

The extension property for partial automorphisms (EPPA)

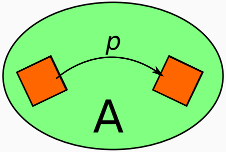
Examples

The following classes have EPPA:

- Sets
- Graphs - Hrushovski 1992
- K_n-free graphs - Herwig 1998
- Generalized to model-theoretic constructions - Herwig, Lascar 2000
- Metric spaces with rational distances - Solecki 2005

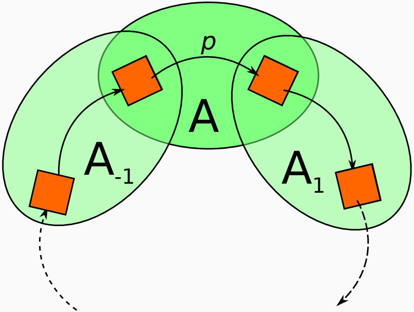
Linear orders and partial orders do not have EPPA.

Motivation for EPPA



Let $(A, d) \in \mathcal{A}^{\delta}_{K,C}$ and p be a partial isomorphism

Motivation for EPPA



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We can form the infinte extension $\bigcup_{i \in \mathbb{Z}} A_i$

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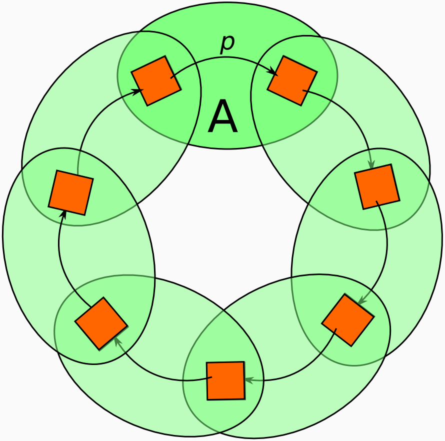

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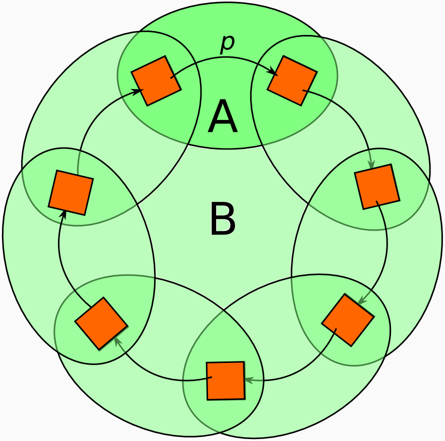
Motivation for EPPA



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For a finite extension, we have to identify $A_n = A_0$

If n big enough, we can complete to $(B, d) \in \mathcal{A}^{\delta}_{K,C}$

EPPA by a result of Herwig, Lascar

Theorem (Herwig, Lascar ’00)

Let $\mathcal{O}$ be a finite set of relational structures, and let $\mathrm{Forb}(\mathcal{O})$ be the class of all structures that contain no homomorphic images of structures in $\mathcal{O}$. Then $\mathrm{Forb}(\mathcal{O})$ has EPPA.

EPPA by a result of Herwig, Lascar

Theorem (Herwig, Lascar '00)

Let $\mathcal{O}$ be a finite set of relational structures, and let $\mathrm{Forb}(\mathcal{O})$ be the class of all structures that contain no homomorphic images of structures in $\mathcal{O}$. Then $\mathrm{Forb}(\mathcal{O})$ has EPPA.

Consequently $\mathcal{A}^{\delta}_{K,C}$ has EPPA:

Let $\mathcal{O}$ be set of obstacles (finitely many cycles) for completion $\mathcal{A}^{\delta}_{K,C}$.
For $(A,d) \in \mathcal{A}^{\delta}_{K,C}$, form an EPPA-witness $(B,d) \in \mathrm{Forb}(\mathcal{O})$. Then:
$(B,d^M) \in \mathcal{A}^{\delta}_{K,C}$ and $\mathrm{Aut}(B,d) \leq \mathrm{Aut}(B,d^M)$.

Results

Theorem AB-WHKKKP '17

Let $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ be an amalgamation class in Cherlin's catalogue.

Then

1. $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ has EPPA, canonical amalgamation and its expansion by linear orders has the Ramsey property,
2. or we are in one of two extremal cases.

Results

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Then

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2. or we are in one of two extremal cases.

Remark

These properties imply several facts about the Fraïssé limits of $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ and their automorphism groups.

Thank you!