Maximal-closed subgroups of Sym(ω) via Henson digraphs

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Outline

1. Closed subgroups of Sym(ω)
2. Homogeneous graphs & Henson digraphs
3. Reducts of Henson digraphs
4. 2^ω many maximal-closed subgroups of Sym(ω)

Closed subgroups of Sym(ω)

Sym(ω) is a topological group with the basis of clopen subgroups:

$\{f \in \mathrm{Sym}(\omega) : f \restriction_A = id \restriction_A\}$ for all finite $A \subset \omega$.

There are 2^ω closed subgroups of Sym(ω).
A closed subgroup $\Sigma < \mathrm{Sym}(\omega)$ is maximal-closed if there is no closed Σ' with $\Sigma < \Sigma' < \mathrm{Sym}(\omega)$.

Question (Macpherson)

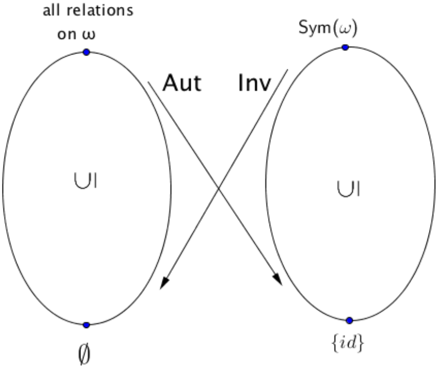
Are there 2^ω non-conjugate (non-isomorphic) maximal-closed subgroups of Sym(ω)?

Conjugation describes permutations up to renaming of elements:

If $f = \kappa^{-1} g \kappa$ then $y = f(x) \leftrightarrow \kappa(y) = g(\kappa(x))$

Aut-Inv Galois connection

Aut($\mathcal{A}$): automorphisms of relational structure $\mathcal{A} = (\omega; R_1, R_2, \ldots)$
Inv(F): relations preserved by $F \subseteq \mathrm{Sym}(\omega)$



Aut(Inv(F)): Minimal closed group containing F

$\mathcal{A}$ is reduct of $\mathcal{A}'$ or $\mathcal{A} \leq_{f.o.} \mathcal{A}'$ if every relation in $\mathcal{A}$ is definable in $\mathcal{A}'$.

$\mathcal{A} \leq_{f.o.} \mathcal{A}' \to \mathrm{Aut}(\mathcal{A}) \geq \mathrm{Aut}(\mathcal{A}')$.

Reducts

Example

$\mathcal{A} = (\omega, \{c\})$... countable set with a constant c.
$\mathrm{Aut}(\mathcal{A}) = \{f \in \mathrm{Sym}(\omega) : f(c) = c\}$. $\mathcal{A}$ has only trivial reducts,
$\mathrm{Aut}(\mathcal{A})$ is maximal-closed subgroup of $\mathrm{Sym}(\omega)$.

Let $(\mathbb{Q}, <)$ be the natural order on the rational numbers.

Cameron '76

There are exactly 5 reducts of $\mathbb{Q}$:

$$\begin{tikzcd}[column sep=small] & (\mathbb{Q},<) \arrow[dl, dash] \arrow[dr, dash] & \\ (\mathbb{Q},\mathrm{Betw}) \arrow[dr, dash] & & (\mathbb{Q},\mathrm{Cycl}) \arrow[dl, dash] \\ & (\mathbb{Q},\mathrm{Sept}) \arrow[d, dash] & \\ & (\mathbb{Q},=) & \end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small] & \mathrm{Sym}(\mathbb{Q}) \arrow[d, dash] & \\ & \langle -_{\mathbb{Q}}, sw_{\mathbb{Q}} \rangle \arrow[dl, dash] \arrow[dr, dash] & \\ \langle sw_{\mathbb{Q}} \rangle \arrow[dr, dash] & & \langle -_{\mathbb{Q}} \rangle \arrow[dl, dash] \\ & \mathrm{Aut}(\mathbb{Q},<) & \end{tikzcd}$$

Oligomorphic permutation groups

A subgroup $\Sigma \leq \mathrm{Sym}(\omega)$ is oligomorphic, if for every $n \in \omega$ the action $\Sigma \curvearrowright \omega^n$ has only finitely many orbits.

$\mathrm{Aut}(\omega, \{c\})$, $\mathrm{Aut}(\mathbb{Q}, <)$ are oligomorphic

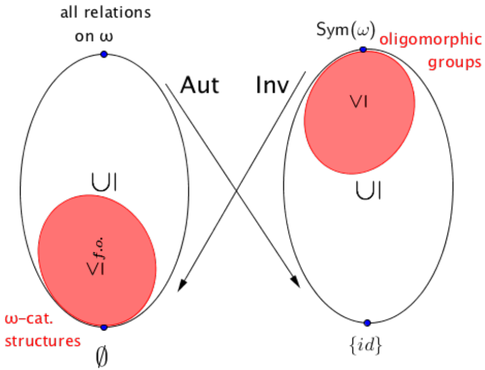
Oligomorphic groups are "big" $\rightarrow$ good candidates for maximal-closed subgroups.

Engeler, Ryll-Nardzewski, Svenonius '59

A closed group $\Sigma \leq \mathrm{Sym}(\omega)$ is oligomorphic, if and only if $\Sigma = \mathrm{Aut}(\mathcal{A})$ for an ω-categorical structure $\mathcal{A}$.
Invariants of $\Sigma \curvearrowright \omega^n$ are exactly the definable relations in $\mathcal{A}$.

Aut-Inv Galois connection

Aut($\mathcal{A}$): automorphisms of relational structure $\mathcal{A} = (\omega; R_1, R_2, \ldots)$
Inv(F): relations preserved by $F \subseteq$ Sym(ω)



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$\mathcal{A} \leq_{f.o.} \mathcal{A}' \rightarrow \mathrm{Aut}(\mathcal{A}) \geq \mathrm{Aut}(\mathcal{A}')$.

For ω-categorical $\mathcal{A}'$ structures: $\mathcal{A} \leq_{f.o.} \mathcal{A}' \leftrightarrow \mathrm{Aut}(\mathcal{A}) \geq \mathrm{Aut}(\mathcal{A}')$.

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Homogeneous structures

A structure is called homogeneous, if every partial isomorphism between finite substructures extends to an automorphism.

Example

The Random graph $(R, \bar{E})$ is the unique countable graph that:

- embeds all finite graphs and
- is homogeneous.

$\mathrm{Aut}(R, \bar{E})$ is oligomorphic.
There are exactly 5 supergroups containing $\mathrm{Aut}(R, \bar{E})$.

Thomas' conjecture

Every countable homogeneous structure in a finite relational language has only finitely many reducts.

Henson graphs

Let $n > 1$. There is a unique countable graph $(H_n, \bar{E})$:

- Finite substructures of $(H_n, \bar{E})$ = graphs not containing the complete graph $(K_n, \bar{E})$,
- $(H_n, \bar{E})$ is homogeneous.

$(H_n, \bar{E})$ is called a Henson graph.

Thomas '91

For every $n > 1$, $(H_n, \bar{E})$ has only trivial reducts and so $\mathrm{Aut}(H_n, \bar{E})$ is maximal-closed in $\mathrm{Sym}(H_n)$.

Also $\mathrm{Aut}(H_n, \bar{E})$ are pairwise non conjugate.

Henson digraphs

Let be a set of finite tournaments. Then there is a unique countable digraph (D_T, E):

- The finite substructures of (D_T, E) are exactly the digraphs omitting T,
- (D_T, E) is homogeneous.

Let $T \neq \emptyset$ and not contain the 2-tournament. Then (D_T, E) is called a Henson digraph.

There are 2^ω non-isomorphic Henson digraphs.

Question

What are the reducts for a given Henson digraph (D_T, E)?

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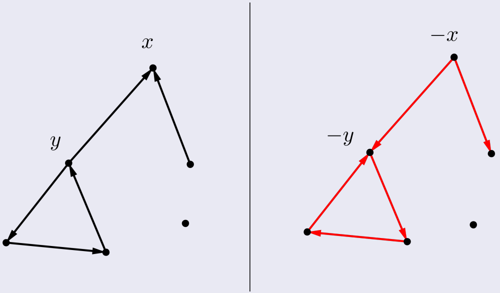
Possible reducts

Example

Let T be closed under switching the direction of all edges. Then there is a bijection $- : D_T \to D_T$ such that $E(x, y) \leftrightarrow E(-y, -x)$.

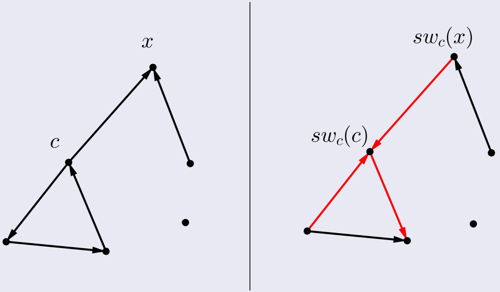


We write $\langle - \rangle$ for the closed group generated by $\mathrm{Aut}(D_T, E) \cup \{-\}$. $\langle - \rangle$ is a proper reduct of $\mathrm{Aut}(D_T, E)$.

Possible reducts

Example

Assume there is a bijection $sw_c : D_T \to D_T$ that switches all edges adjacent to a vertex c, while preserving all other edges:



Then $\langle sw_c \rangle$ is a proper reduct.
The existence of sw_c only depends on T.

Canonical functions

A function $f : \mathcal{A} \to \mathcal{B}$ is called canonical, if it maps n-orbits of $\mathrm{Aut}(\mathcal{A})$ to n-orbits of $\mathrm{Aut}(\mathcal{B})$.

Example

$- : (D_T, E) \to (D_T, E)$ is canonical.
$sw_c : (D_T, E) \to (D_T, E)$ is not canonical, but
$sw_c : (D_T, E, c) \to (D_T, E)$ is canonical.

Bodirsky, Pinsker, Tsankov '13

Let $(D, E, <)$ be a Henson ordered digraph. Let $f \in \mathrm{Sym}(D)$ and $c_1, \ldots, c_n \in D$. Then there exists a function $g : D \to D$ such that

- g lies in the topological closure of $\langle \mathrm{Aut}(D, E) \cup \{f\} \rangle$ in D^D,
- $g(c_i) = f(c_i)$ for $i = 1, \ldots n$,
- $g : (D, E, <, c_1, \ldots, c_n) \to (D, E)$ is canonical.

There are only finitely many "behaviours" of canonical functions.

The reducts of Henson digraphs

Theorem (Agarwal, MK '15)

Let (D_T, E) be a Henson digraph and $G \geq \mathrm{Aut}(D_T, E)$.
Let $\bar{E}(x, y) \leftrightarrow E(x, y) \vee E(y, x)$. Then

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- $G < \mathrm{Aut}(D_T, \bar{E})$ or $G \geq \mathrm{Aut}(D_T, \bar{E})$

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- If $G < \mathrm{Aut}(D_T, \bar{E})$, then
  $G = \mathrm{Aut}(D_T, E)$, $\langle - \rangle$, $\langle sw \rangle$ or $\langle -, sw \rangle$

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- One of the following holds
    - $(D_T, \bar{E})$ is the Random graph

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  - $(D_T, \bar{E})$ is a Henson graph

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- $G < \mathrm{Aut}(D_T, \bar{E})$ or $G \geq \mathrm{Aut}(D_T, \bar{E})$
- If $G < \mathrm{Aut}(D_T, \bar{E})$, then $G = \mathrm{Aut}(D_T, E)$, $\langle - \rangle$, $\langle sw \rangle$ or $\langle -, sw \rangle$
- One of the following holds
    - $(D_T, \bar{E})$ is the Random graph
    - $(D_T, \bar{E})$ is a Henson graph
    - $(D_T, \bar{E})$ has no proper reducts and $\mathrm{Aut}(D_T, \bar{E}) = \max\{\mathrm{Aut}(D_T, E), \langle - \rangle, \langle sw \rangle, \langle -, sw \rangle\}$

The reducts of Henson digraphs

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Sym}(D_T) \arrow[d, no head] & \\
 & \langle sw_{\bar{E}}, -_{\bar{E}} \rangle \arrow[dl, no head] \arrow[dr, no head] & \\
\langle sw_{\bar{E}} \rangle \arrow[dr, no head] & & \langle -_{\bar{E}} \rangle \arrow[dl, no head] \\
 & \mathrm{Aut}(D_T, \bar{E}) \arrow[d, no head] & \\
 & \langle sw, - \rangle \arrow[dl, no head] \arrow[dr, no head] & \\
\langle sw \rangle \arrow[dr, no head] & & \langle - \rangle \arrow[dl, no head] \\
 & \mathrm{Aut}(D_T, E) &
\end{tikzcd}$$

The lattice to the left shows all potential reducts of a given Henson digraph (D_T, E).

- Always finitely many reducts
- $T' \supset T$ does not imply that $D_{T'}$ has less reducts that D_T.

(D_T, E) has only trivial reducts if

- T not closed under $-$ and sw
- $(D_T, \bar{E})$ is not homogeneous

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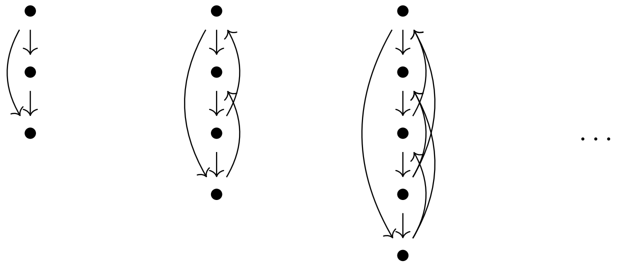
2^ω non-isomorphic Henson digraphs

Let I_n be the tournament we obtain by taking a linear order of size n and flipping all edges $(i, i+1)$ for $i < n$ and $(1, n)$.



I_3 I_4 I_5

The Henson digraphs (D_T, E) for $T \subseteq \{I_n : n > 6\}$ are pairwise non-isomorphic. But $(D_T, \bar{E})$ is the random graph...

2^ω Henson digraphs with trivial reducts

There is a tournament X that has a sink, but no source and is not embeddable in any I_n. Take

$$T = \{X' : |X'| = |X| + 1, X \subset X'\} \cup T' \text{ for } T' \subseteq \{I_n : n > |X| + 1\}$$

Then the induced Henson digraph has only trivial reducts.

Two such Henson digraphs are not isomorphic.

It is easy to see that their automorphism groups are non-conjugate.

Reconstruction

Rubin '87

For two Henson digraphs (D_1, E) and (D_2, E) the following are equivalent:

- $\mathrm{Aut}(D_1, E)$ and $\mathrm{Aut}(D_2, E)$ are conjugate
- $\mathrm{Aut}(D_1, E) \cong_T \mathrm{Aut}(D_2, E)$
- $\mathrm{Aut}(D_1, E) \cong \mathrm{Aut}(D_2, E)$

Conclusion

There are 2^ω non-isomorphic maximal-closed subgroups of $\mathrm{Sym}(\omega)$.

Thank you!