Constraint satisfaction problems over infinite domains

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Schaefer's theorem

Let Φ be a set of propositional formulas.

Boolean-SAT(Φ)

Input:

- A set of propositional variables V and
- statements $\phi_1, \ldots, \phi_n$ about the variables taken from Φ

Problem:

Is $\phi_1 \wedge \ldots \wedge \phi_n$ satisfiable?

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Schaefer '78 (1661 citations on Google scholar!)

Boolean-SAT(Φ) is either in P or in NP-complete, for all Φ.

Schaefer's theorem for partial orders

Let Φ be a finite set of quantifier-free $\leq$-formulas.

Poset-SAT(Φ)

Input:

- A set of variables V and
- statements $\phi_1, \ldots, \phi_n$ about the variables taken from Φ

Problem:

Is there a partial order that satisfies $\phi_1 \wedge \ldots \wedge \phi_n$?

Computational complexity is in NP and depends on Φ.

Theorem (MK, TVP '16)

Poset-SAT(Φ) is either in P or in NP-complete, for all Φ.

Outline

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1. **Constraint satisfaction problems**
2. The universal algebraic approach
3. Poset-SAT
4. Summary

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Informally in a constraint satisfaction problem or CSP the aim is to check if there are objects that satisfy a given set of constraints (e.g. Sudoku, Time scheduling, system of linear equations).

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Question

Given Γ, what is the computational complexity of $\mathrm{CSP}(\Gamma)$?

Boolean-SAT

2-SAT

Instance: A set of 2-clauses (x, y)
Problem: Is there a satisfying truth assignment?

$\mathrm{CSP}(\{0,1\}; \mathrm{2OR}, \mathrm{NEQ})$ with
$\mathrm{2OR} = \{(1,1),(0,1),(1,0)\}$ and $\mathrm{NEQ} = \{(0,1),(1,0)\}$.

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Positive 1-3-SAT

Instance: 3-clauses (x, y, z) with positive literals
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CSPs over $\{0,1\}$ are exactly the Boolean-SAT(Φ) problems.

More examples

$\mathrm{CSP}(\mathbb{Q}, <)$

Instance: A pp-sentence in the language $<$
Problem: Does it hold in $(\mathbb{Q}, <)$?

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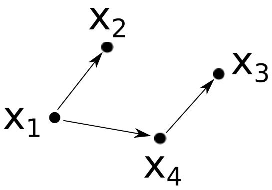

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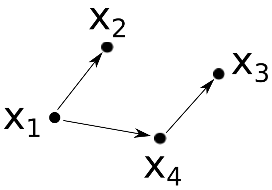
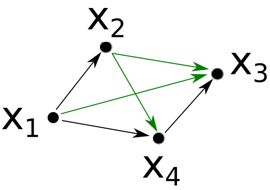


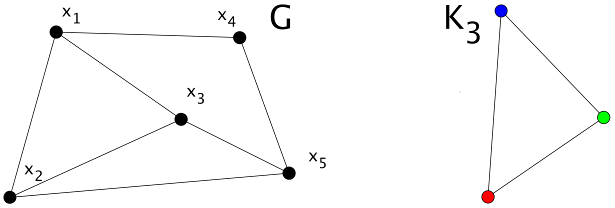


More examples

3-COLOR

Instance: A finite graph $(G; E)$
Problem: Is it colorable with 3-colors?

CSP with template (K_3, E)



Instance: $\exists x_1, \ldots, x_5 \; E(x_1, x_2) \wedge E(x_1, x_4) \wedge \cdots \wedge E(x_4, x_5)$

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If Γ is finite, $\mathrm{CSP}(\Gamma)$ is in NP.

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- If $|\Gamma| \geq 4$: ...?

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For structures Γ, Δ write $\Gamma \leq_{pp} \Delta$ if every relation in Γ has a definition with primitive positive formulas in Δ.

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Essential observation

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We only need to study structures up to pp-interdefinable.

Polymorphism clones

We say a function $f : D^n \to D$ preserves a relation $R \subseteq D^k$ if for all $\bar{r}_1, \ldots, \bar{r}_n \in R$ also $f(\bar{r}_1, \ldots, \bar{r}_n) \in R$.

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A function $f : \Gamma^n \to \Gamma$ is called a polymorphism if it preserves all relations in Γ.

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Theorem (Geiger '68)

For finite structures Δ and Γ:

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$\to$ the complexity of $\mathrm{CSP}(\Gamma)$ is determined by $\mathrm{Pol}(\Gamma)$!

Schaefer's theorem revisited

The Boolean $\mathrm{CSP}(\Gamma)$ is in P if and only if

All relations in Γ	$\mathrm{Pol}(\Gamma)$ contains
contain $(0,\ldots,0)$	constant 0
contain $(1,\ldots,1)$	constant 1
are Horn	$(x,y) \to x \wedge y$
are dual Horn	$(x,y) \to x \vee y$
are affine	$(x,y,z) \to x - y + z$
are 2-clauses	$(x,y,z) \to (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$

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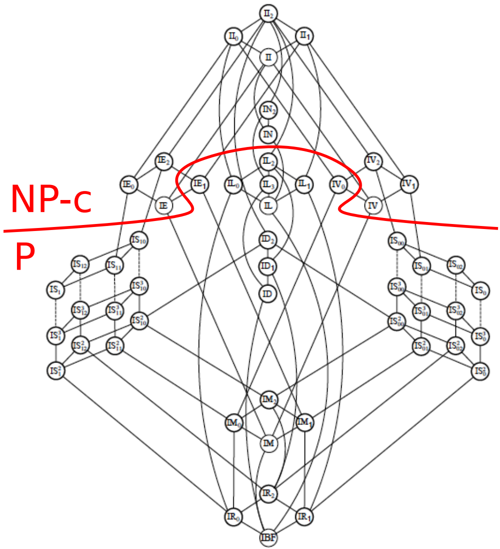
Tractability conjecture (Bulatov, Jeavons, Krokhin,...)

Let Γ be finite (+ mc core, contains all constants). Then either

- $\exists f \in \mathrm{Pol}(\Gamma) : f(x_1, x_2, \dots, x_n) = f(x_2, x_3, \dots, x_n, x_1)$ and $\mathrm{CSP}(\Gamma)$ is in P
- or $\mathrm{CSP}(\Gamma)$ is NP-complete.

The lattice of all clones on $\{0, 1\}$



Infinite CSPs

If Γ is infinite, $\mathrm{CSP}(\Gamma)$ can be undecidable:

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Instance: Equations using $0, 1, +, \cdot$
Problem: Is there an integer solution?

$\mathrm{CSP}(\mathbb{Z}; 0, 1, +, \cdot)$.

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Hope: Algebraic approach still works for "nice" structures.

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1. Constraint satisfaction problems
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Poset-SAT as CSP

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For every $\{\leq\}$-formula $\phi(x_1, \ldots, x_n)$ we define the relation

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$$\text{Poset-SAT}(\Phi) = \mathrm{CSP}((P; R_\phi)_{\phi \in \Phi}).$$

$(P; R_\phi)_{\phi \in \Phi}$ is a reduct of $\mathbb{P}$, i.e. a structure that is first-order definable in $\mathbb{P}$.

CSPs over random partial order

ω-categorical structure

A structure Γ is called ω-categorial, if its theory has, up to isomorphism, exactly one countable model.

Engeler, Ryll-Nardzewski, Svenonius

An countably infinite structure Γ with countable signature is ω-categorical if and only if for every $k \in \mathbb{N}$, there are finitely many k-orbits of $\mathrm{Aut}(\Gamma)$.

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Why is $\mathbb{P}$ ω-categorical?

For every $k \in \mathbb{N}$, there are finitely many posets on k elements.

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Bodirsky, Nešetřil '03

For ω-categorical structures Γ, Δ we have

$$\Gamma \leq_{pp} \Delta \leftrightarrow \mathrm{Pol}(\Gamma) \supseteq \mathrm{Pol}(\Delta)$$

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Strategy for Poset-SAT(Φ)

Boolean-SAT(Φ)	Poset-SAT(Φ)
$\downarrow$	$\downarrow$
CSPs of Boolean structures $(\{0,1\}; R_1, \ldots R_n)$ are reducts of $(\{0,1\}, 0, 1)$	CSPs of reducts of random partial order $\mathbb{P}$
$\downarrow$	$\downarrow$
Clones over $\{0,1\}$	Closed clones containing $\mathrm{Aut}(\mathbb{P})$

Important NP-complete relations

- $\mathrm{Betw}(x, y, z) := x < y < z \vee z < y < x$.
- $\mathrm{Cycl(x, y, z)} := (x < y \wedge y < z) \vee (z < x \wedge x < y) \vee (y < z \wedge z < x) \vee (x < y \wedge z \perp x \wedge z \perp y) \vee (y < z \wedge x \perp y \wedge x \perp z) \vee (z < x \wedge y \perp z \wedge y \perp x)$.
- $\mathrm{Sep}(x, y, z, t) := ((\mathrm{Cycl}(x, y, z) \wedge \mathrm{Cycl}(y, z, t) \wedge \mathrm{Cycl}(x, y, t) \wedge \mathrm{Cycl}(x, z, t)) \vee (\mathrm{Cycl}(z, y, x) \wedge \mathrm{Cycl}(t, z, y) \wedge \mathrm{Cycl}(t, y, x) \wedge \mathrm{Cycl}(t, z, x))$.
- $\mathrm{Low}(x, y, z) := (x < y \wedge x \perp z \wedge y \perp z) \vee (x < z \wedge x \perp y \wedge z \perp y)$.

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Theorem (MK, TVP '16)

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- Low, Betw, Cycl or Sep is pp-definable in Γ and $\mathrm{CSP}(\Gamma)$ is NP-complete.
- $\mathrm{Pol}(\Gamma)$ contains functions f, g_1, g_2 such that

$$g_1(f(x,y)) = g_2(f(y,x))$$

and $\mathrm{CSP}(\Gamma)$ can be solved in polynomial time.

Consequence:

Poset-SAT(Φ) is in P or NP-complete.

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Consequence:

Poset-SAT(Φ) is in P or NP-complete.
Given Φ, it is decidable to tell if Poset-SAT(Φ) is in P.

The method for the classification

Canonicalization theorem (Bodirsky, Pinsker and Tsankov, 2012)

Let Δ be ordered homogeneous Ramsey with finite relational signature, $f : \Delta \to \Delta$, and let $c_1, c_2, \ldots, c_n \in \Delta$. Then f generates over Δ a function which agrees with f on $\{c_1, c_2, \ldots, c_n\}$ and which is canonical as a function from $(\Delta, c_1, c_2, \ldots, c_n)$.

The method for the classification

Canonical functions

A function $f : P^2 \to P$ is called canonical if the type of image depends only on the types of arguments of the function in the domain.

Example

$e_<$	$=$	$<$	$>$	$\perp$
$=$	$=$	$\perp$	$\perp$	$\perp$
$<$	$\perp$	$<$	$\perp$	$\perp$
$>$	$\perp$	$\perp$	$>$	$\perp$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

Embedding from $(P; <)^2$ to $(P; <)$.

$e_\leq$	$=$	$<$	$>$	$\perp$
$=$	$=$	$<$	$>$	$\perp$
$<$	$<$	$<$	$\perp$	$\perp$
$>$	$>$	$\perp$	$>$	$\perp$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

Embedding from $(P; \leq)^2$ to $(P; \leq)$.

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$=$	$=$	$<$	$>$	$\perp$
$<$	$<$	$<$	$\perp$	$\perp$
$>$	$>$	$\perp$	$>$	$\perp$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

Embedding from $(P; \leq)^2$ to $(P; \leq)$.

for every $x < y$ and $x' > y'$, we have $e_<(x, x') \perp e_<(y, y')$.

The method for the classification

Lemma

Let Γ be a reduct of $(P;\leq)$. If $<,\perp \in \langle\Gamma\rangle_{pp}$, $\mathrm{Low} \notin \langle\Gamma\rangle_{pp}$, then $e_<$ or $e_\leq$ is a polymorphism of Γ.

Proof

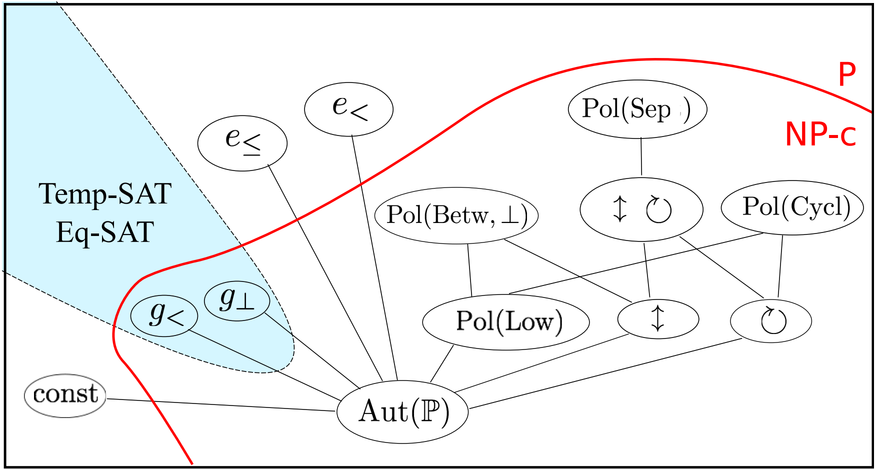
1. Since Low is not primitive positive definable in Γ, there is a binary polymorphism f of Γ that violates Low.
2. We can find three elements $a, b, c \in P$ such that $a < b \wedge ab\perp c$, and $(f(a,a), f(b,c), f(c,b)) \notin \mathrm{Low}$.
3. We can assume that f is canonical as a function from $(P;\leq,\preceq,a,b,c)^2$ to $(P;\leq,\preceq,a,b,c)$.
4. Use an extensive combinatorial analysis on f...

The method for classification

Using the same method one could successfully classify the complexity of a number of CSPs on infinite domains.

1. Graph-SAT (M. Bodirsky and M. Pinsker, 2015).
2. Phylogeny CSPs (M. Bodirsky, P. Jonsson and T. V. Pham, 2015).
3. Henson graphs (M. Bodirsky, B. Martin, M. Pinsker and A. Pongrácz, 2016).
4. Semilinear order-SAT (M. Bodirsky and T. V. Pham, in preparation).

Lattice of polymorphism clones containing $\mathrm{Aut}(\mathbb{P})$



Thank you!