

CSP lecture 24/25 – Problem Set 1

$\mathbb{A} = (A; R_1, R_2, \dots)$ is called a *relational structure* if

- A is a set, called *domain*,
- $R_1, R_2, \dots$ are relations on A , i.e. each $R_i \in \Gamma$ is a subset of A^{n_i} for some n_i ,

CSP($\mathbb{A}$)

Given a list of constraints $R_i(x_{i_1}, \dots, x_{i_r}), R_j(x_{j_1}, \dots, x_{j_s}), R_k(x_{k_1}, \dots, x_{k_t}), \dots$

Decide whether they are satisfiable.

Consider the following relations on $\{0, 1\}$:

- $C_i := \{i\}$, for $i \in \{0, 1\}$
- $R := \{(0, 0), (1, 1)\}$
- $N := \{(0, 1), (1, 0)\}$
- $S_{ij} := \{0, 1\}^2 \setminus \{(i, j)\}$, for $i, j \in \{0, 1\}$
- $H := \{0, 1\}^3 \setminus \{(1, 1, 0)\}$
- $G_1 := \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}$, $G_2 := \{(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)\}$

Problem 1. Find a polynomial-time algorithm for CSP($\mathbb{A}$), where

1. $\mathbb{A} = (\{0, 1\}; R)$
2. $\mathbb{A} = (\{0, 1\}; R, C_0, C_1)$
3. $\mathbb{A} = (\{0, 1\}; S_{10})$
4. $\mathbb{A} = (\{0, 1\}; S_{10}, C_0, C_1)$
5. $\mathbb{A} = (\{0, 1\}; S_{01}, S_{10}, C_0, C_1)$
6. $\mathbb{A} = (\{0, 1\}; N)$
7. $\mathbb{A} = (\{0, 1\}; R, N, C_0, C_1)$
8. $\mathbb{A} = (\{0, 1\}; R, N, C_0, C_1, S_{00}, S_{01}, S_{10}, S_{11})$
9. $\mathbb{A} = (\{0, 1\}; \text{all unary and binary relations})$

Problem 2. Find a polynomial-time algorithm for CSP($\{0, 1\}; H, C_0, C_1$).

Problem 3. Find a polynomial-time algorithm for CSP($\{0, 1\}; C_0, C_1, G_1, G_2$).

Problem 4. Find a polynomial-time algorithm for CSP($\mathbb{Q}; <$).

CSP lecture 24/25 – Problem Set 2

The *type* of a relational structure $(A; R_1, \dots, R_s)$ is the tuple $(\text{ar}(R_1), \dots, \text{ar}(R_s))$, where $\text{ar}(R)$ is the arity of the relation R .

Suppose the type of $\mathbb{A} = (A; R_1, \dots, R_t)$ and $\mathbb{B} = (B; S_1, \dots, S_t)$ is $(n_1, \dots, n_t)$. A mapping $\phi : A \rightarrow B$ is called a *homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ if $(a_1, \dots, a_{n_i}) \in R_i \Rightarrow (\phi(a_1), \dots, \phi(a_{n_i})) \in S_i$ for every i . If such a homomorphism exists we write $\mathbb{A} \rightarrow \mathbb{B}$. A homomorphism $\mathbb{A} \rightarrow \mathbb{A}$ is an *endomorphism*, a bijective endomorphism is an *automorphism*.

Hom($\mathbb{A}$)
Given a finite relational structure $\mathbf{X}$ of the same type as $\mathbb{A}$. Decide whether $\mathbf{X} \rightarrow \mathbb{A}$.

Problem 1. Find a polynomial algorithm for Hom($\mathbb{A}$) where

1. $\mathbb{A} = (\{0, 1\}; N)$ (notation is from the 1st problem set)
2. $\mathbb{A} = (\{0, 1\}; N, C_0, C_1)$ (notation is from the 1st problem set)
3. $\mathbb{A} = (\{0, 1\}; S_{00}, S_{11})$ (notation is from the 1st problem set)

Recall that a decision problem $\mathcal{P}_1$ is *polynomially reducible* to $\mathcal{P}_2$ if there exists a polynomial-time algorithm that transforms an input I of $\mathcal{P}_1$ to an input $r(I)$ of $\mathcal{P}_2$ so that I is a Yes-instance iff $r(I)$ is a Yes-instance. In such a case, we write $\mathcal{P}_1 \leq_P \mathcal{P}_2$. When $\mathcal{P}_1 \leq_P \mathcal{P}_2 \leq_P \mathcal{P}_1$, we write $\text{CSP}(\mathbb{A}) \sim_P \text{CSP}(\mathbb{B})$ and say that the two problems are *polynomially equivalent*.

Problem 2. $\mathbb{A} = (\{0, 1, 2\}; N)$, where $N = \{0, 1, 2\}^2 \setminus \{(0, 0), (1, 1), (2, 2)\}$. Prove that $\text{CSP}(\mathbb{A})$ is polynomially equivalent to Hom($\mathbb{A}$).

Problem 3. $\mathbb{A}$ is a relational structure. Prove that $\text{CSP}(\mathbb{A})$ is polynomially equivalent to Hom($\mathbb{A}$).

Observe that if $\text{CSP}(\mathbb{A}) \leq_P \text{CSP}(\mathbb{B})$ and $\text{CSP}(\mathbb{B})$ is in P (i.e., solvable in polynomial time), then $\text{CSP}(\mathbb{A})$ is in P. Similarly, if $\text{CSP}(\mathbb{A}) \leq_P \text{CSP}(\mathbb{B})$ and $\text{CSP}(\mathbb{A})$ is NP-complete, then $\text{CSP}(\mathbb{B})$ is NP-complete.

Problem 4. Prove that $\text{CSP}(\mathbb{A}) \sim_P \text{CSP}(\mathbb{B})$, where

- $\mathbb{A} = (\{0, 1, 2\}; C_0, C_1, Q)$, where

$$C_0 = \{0\}, C_1 = \{1\}, Q = \{000, 110, 120, 210, 101, 102, 201, 202, 011, 012, 021\}$$

(Q is a ternary relation, we omit the commas and parentheses, eg. 110 stands for $(1, 1, 0)$.)

- $\mathbb{B} = (\{0, 1\}; C_0, C_1, G_1)$ (where the notation is from the 1st problem set).

Hint: use homomorphisms $\mathbb{A} \rightarrow \mathbb{B}$ and $\mathbb{B} \rightarrow \mathbb{A}$.

Problem 5. Prove that for each finite relational structure $\mathbb{A}$ there exists a relational structure $\mathbb{B}$ such that

- there exists a homomorphism $\mathbb{A} \rightarrow \mathbb{B}$ and a homomorphism $\mathbb{B} \rightarrow \mathbb{A}$, and
- $\mathbb{B}$ is a *core*, that is, each endomorphism of $\mathbb{B}$ is an automorphism.

Problem 5.1. Deduce that we can WLOG concentrate on CSPs over cores.

Problem 5.2. Prove that such a core is unique up to isomorphism.

Problem 5.3. Find a relational structure $\mathbb{A}$ such that every structure $\mathbb{B}$ with homomorphisms $\mathbb{A} \rightarrow \mathbb{B}$ and $\mathbb{B} \rightarrow \mathbb{A}$ is *not* a core. Hint: $\mathbb{A}$ can be taken to be a directed graph.

Problem 6. Suppose

- $\mathbb{A} = (A; R_1, R_2, R_4)$ is a relational structure, where each R_i is an i -ary relation.
- E is the equality relation, i.e. $E = \{(a, a) : a \in A\}$
- S is the ternary relation on A defined by

$$S(x, y, z) = R_1(x) \wedge R_2(x, z) \wedge R_4(y, z, y, x)$$

- T is the binary relation defined by $T(x, y) = (\exists z \in A) S(x, y, z)$

Prove that

1. $\text{CSP}(A; R_1, R_2, R_4, E) \leq_P \text{CSP}(\mathbb{A})$
2. $\text{CSP}(A; R_1, R_2, R_4, E, S) \leq_P \text{CSP}(\mathbb{A})$
3. $\text{CSP}(A; R_1, R_2, R_4, E, S, T) \leq_P \text{CSP}(\mathbb{A})$

Problem 6.1. Try to formulate a general theorem covering these particular cases.

Problem 7. Prove that

1. $\text{CSP}(\{0, 1, 2\}; C_0, C_1, N) \sim_P \text{CSP}(\{0, 1, 2\}; C_0, C_1, C_2, N)$
2. $\text{CSP}(\{0, 1, 2\}; N) \sim_P \text{CSP}(\{0, 1, 2\}; N')$
3. $\text{CSP}(\{0, 1\}; C_0, C_1, R) \sim_P \text{CSP}(\{0, 1\}; R')$

where

$$\begin{aligned} N &= \{0, 1, 2\}^2 \setminus \{(0, 0), (1, 1), (2, 2)\} & N' &= \{0, 1, 2\}^3 \setminus \{(0, 0, 0), (1, 1, 1), (2, 2, 2)\} \\ R &= \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\} & R' &= \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \end{aligned}$$

Hint: try to use the general theorem from Problem 6.1.

Problem 8. Prove that $\text{CSP}(\mathbb{A})$, $\text{CSP}(\mathbb{B})$ and $\text{CSP}(\mathbb{C})$ are polynomially equivalent, where

$$\begin{aligned} \mathbb{A} &= (\{0, 1, 2\}; C_0, C_1, C_2, N), & N &= \{0, 1, 2\}^2 \setminus \{(0, 0), (1, 1), (2, 2)\} \\ \mathbb{B} &= (\{0, 1\}; S_{000}, S_{001}, S_{011}, S_{111}), & S_{ijk} &= \{0, 1\}^3 \setminus \{(i, j, k)\} \\ \mathbb{C} &= (\{0, 1\}; C_0, C_1, R), & R &= \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\} \end{aligned}$$

Problem 9. Prove that $\text{CSP}(\mathbb{A}) \sim_P \text{CSP}(\{0, 1, 2\}; N)$, where $\mathbb{A}, N$ are from the previous problem.

Problem 10. For each finite relational structure $\mathbb{A}$, find an input of $\text{CSP}(\mathbb{A})$ whose solutions precisely correspond to endomorphisms of $\mathbb{A}$.

Problem 11. Let $\mathbb{A}$ be a finite *core* and let $\mathbb{B}$ be the relational structure formed from $\mathbb{A}$ by adding all the unary relations $C_a = \{a\}$, $a \in A$. Prove that $\text{CSP}(\mathbb{A}) \sim_P \text{CSP}(\mathbb{B})$.

Problem 12. Let $\mathbb{A}$ be a finite relational structure such that $\text{CSP}(\mathbb{A})$ is in P. Prove that there is a polynomial-time algorithm for finding a solution of $\text{CSP}(\mathbb{A})$.

CSP lecture 24/25 – Problem Set 3

An n -ary operation on a set A is a mapping $A^n \rightarrow A$. The n -ary projection onto the i -th coordinate (on a set A) is the operation π_i^n defined by $\pi_i^n(a_1, \dots, a_n) = a_i$ for any $a_1, \dots, a_n \in A$.

An n -ary operation $f : A^n \rightarrow A$ preserves an m -ary relation $R \subseteq A^m$ if $f(\mathbf{r}_1, \dots, \mathbf{r}_n) \in R$ (operation is applied coordinate-wise) whenever $\mathbf{r}_1, \dots, \mathbf{r}_n \in R$. In other words, for any $m \times n$ matrix whose columns are in R , f applied to the rows of this matrix gives a tuple in R . In such a situation, we also say that R is compatible with f , or R is invariant under f , or f is a polymorphism of R .

An operation $A^n \rightarrow A$ is a polymorphism of a relational structure $\mathbb{A} = (A; \dots)$ if it preserves all the relations in $\mathbb{A}$. The set of all polymorphisms of $\mathbb{A}$ is denoted $\text{Pol}(\mathbb{A})$.

Problem 1. Observe that

1. $f : A^n \rightarrow A$ is compatible with every singleton unary relation $\{a\}$, $a \in A$, iff $f(a, \dots, a) = a$ for all $a \in A$;
2. the constant unary operation $c_a : A \rightarrow A$ (defined by $c_a(x) = a$ for any $x \in A$) is compatible with $R \subseteq A^n$ iff R contains the tuple $(a, a, \dots, a)$.

Problem 2. Let A be a set. Prove that f preserves every relation on A if and only if f is a projection.

Problem 3. Let $\mathbb{A} = (A; \dots)$ be a relational structure, $f \in \text{Pol}(\mathbb{A})$ a binary polymorphism and $g \in \text{Pol}(\mathbb{A})$ a ternary polymorphism. Then the 4-ary operation h defined by

$$h(x_1, x_2, x_3, x_4) = g(x_1, f(x_3, g(x_2, x_2, x_4)), x_3)$$

is a polymorphism of $\mathbb{A}$ as well. Try to formulate a general statement.

Problem 4. Find all unary and binary polymorphisms of the structure $\mathbb{A} = (\{0, 1\}; H, C_0, C_1)$ from Problem Set 1 (Problem 2 – HORN-SAT).

Problem 5. Find all unary and binary polymorphisms of the structure

$$\mathbb{A} = (\{0, 1\}; \text{all unary and binary relations})$$

from Problem Set 1 (Problem 1 – 2-SAT). Find some nice nontrivial (= not a projection) polymorphism of $\mathbb{A}$.

Problem 6. Find all unary, binary, and ternary polymorphisms of the structure $\mathbb{A} = (\{0, 1\}; C_0, C_1, G_1, G_2)$ from Problem Set 1 (Problem 3 – LIN-EQ($\mathbb{Z}_2$)).

A relation $R \subseteq A^m$ is *pp-definable* from $\mathbb{A} = (A; \dots)$ if it can be defined from relations in $\mathbb{A}$ by a pp-formula, that is, a formula which only uses conjunction, equality, and existential quantification. A relational structure $\mathbb{B} = (B; \dots)$ is pp-definable from $\mathbb{A}$ if $A = B$ and each relation in $\mathbb{B}$ is pp-definable from $\mathbb{A}$. We also say that $\mathbb{A}$ pp-defines $\mathbb{B}$.

Problem 7. Prove that any relation pp-definable from $\mathbb{A}$ is invariant under every polymorphism of $\mathbb{A}$.

Problem 8. Find all polymorphisms of the structure $\mathbb{B}$ in Problem Set 2 (Problem 8 – 3-SAT). Hint: only projections; possible approach: (1) pp-define the four-ary relations of the form $R_{a,b,c,d} = \{0, 1\}^4 \setminus \{(a, b, c, d)\}$, (2) pp-define all four-ary relations (3) similarly, pp-define every relation, (4) use Problem 2.

Problem 9. Let $\mathbb{A}$ be a finite structure. Prove that a relation invariant under every polymorphism of $\mathbb{A}$ is pp-definable from $\mathbb{A}$. Proof strategy:

- (i) Denote $R = \{(c_{11}, \dots, c_{1k}), \dots, (c_{m1}, \dots, c_{mk})\}$
- (ii) Let $\mathbf{a}_1, \dots, \mathbf{a}_n$ be a complete list of m -tuples of elements of A (i.e. $n = |A|^m$)
- (iii) Prove that the relation

$$S = \{(f(\mathbf{a}_1), \dots, f(\mathbf{a}_n)) : f \text{ is an } m\text{-ary polymorphism}\}$$

is pp-definable from $\mathbb{A}$ (no need to use existential quantification)

- (iv) Existentially quantify over all coordinates but those corresponding to $(c_{11}, \dots, c_{m1}), \dots, (c_{1k}, \dots, c_{mk})$
- (v) Prove that the obtained relation contains R (because of projections) and is contained in R (because of compatibility)

Problem 10. Let $\mathbb{A} = (\mathbb{Z} \times \mathbb{Z}; R, U)$, where

$$R = \{((x, y), (x', y')) \mid x = x', |y' - y| \in \{1, 2\}\}, \quad U = \{(0, 0)\}.$$

Prove that $\{(0, y) \mid y \in \mathbb{Z}\}$ is invariant under every polymorphism of $\mathbb{A}$, but that this set is not pp-definable from $\mathbb{A}$.

Problem 11. Observe that, for finite structures $\mathbb{A}$ and $\mathbb{B}$,

1. $\mathbb{A}$ pp-defines $\mathbb{B}$ iff $\text{Pol}(\mathbb{A}) \subseteq \text{Pol}(\mathbb{B})$ and in such a case $\text{CSP}(\mathbb{B}) \leq_P \text{CSP}(\mathbb{A})$;
2. any CSP over a two-element structure is polynomially reducible to 3-SAT
3. if $\text{Pol}(\mathbb{A}) \subseteq \text{Pol}(\mathbb{B})$, then the proof of Problem 9 gives an explicit pp-formulas defining relations in $\mathbb{B}$ from relations in $\mathbb{A}$.
4. In particular, for $\mathbb{B}$ and $\mathbb{C}$ as in Problem Set 2, Problem 4, we get $\text{CSP}(\mathbb{C}) \leq \text{CSP}(\mathbb{B})$. How large are the explicit formulas defining relations in $\mathbb{C}$ from relations in $\mathbb{B}$?

CSP lecture 24/25 – Problem Set 4

A set of operations on a set A is a (*function*) *clone* on A if it contains all projections and is closed under composition (as in Problem 3, Problem Set 3). A function clone on A is called *idempotent* if for every operation f in it and every $a \in A$, $f(a, a, \dots, a) = a$. For a se

Problem 1. Recall that for any relational structure $\mathbb{A}$, $\text{Pol}(\mathbb{A})$ is a clone.

In this problem set, we focus on function clones on the set $A = \{0, 1\}$. We use the following notation for some special operations on $\{0, 1\}$:

$\wedge$ the binary minimum operation

$\vee$ the binary maximum operation

maj the ternary majority operation defined by $\text{maj}(a, a, b) = \text{maj}(a, b, a) = \text{maj}(b, a, a) := a$ for every $a, b \in \{0, 1\}$

min the ternary minority operation defined by $\text{min}(a, a, b) = \text{min}(a, b, a) = \text{min}(b, a, a) := b$ for every $a, b \in \{0, 1\}$

An operation $f : A^n \rightarrow A$ is called *essentially unary* if there exist i and a unary operation $\alpha : A \rightarrow A$ such that $f(x_1, \dots, x_n) = \alpha(x_i)$ for every $x_1, \dots, x_n \in A$.

Problem 2. Assume that $\mathcal{A}$ is an idempotent clone on $A = \{0, 1\}$ that contains neither $\wedge$ nor $\vee$. Show that the only binary operations in $\mathcal{A}$ are the two projections.

Problem 3. Assume that $\mathcal{A}$ is an idempotent clone on $A = \{0, 1\}$ that contains neither of the operations $\wedge, \vee, \text{maj}, \text{min}$. Show that the only binary and ternary operations in $\mathcal{A}$ are the projections.

Problem 4. Assume that $\mathcal{A}$ is an idempotent clone on $A = \{0, 1\}$ that contains neither of the operations $\wedge, \vee, \text{maj}, \text{min}$. Show that $\mathcal{A}$ contains only projections.

Hint: possible strategy

- Let $f \in \mathcal{A}$ be n -ary with $n \geq 4$.
- Assume first $f(1, 0, 0, \dots, 0) = 1$. Use the binary operation $g(x, y) := f(x, y, \dots, y)$ to show that $f(0, 1, \dots, 1) = 0$. Use ternary operations of the form $g(x, y, z) := f(w_1, w_2, \dots)$ where $w_1, w_2, \dots \in \{x, y, z\}$ to show that f is the projection onto the first coordinate.
- Deduce that if f is not a projection, then $f(x, \dots, x, y, x, \dots, x) = x$ for every x, y and every position of y .
- Assuming this and using appropriate ternary operations (similar as above) show that $f(x, \dots, x, y, y) = x, \dots$, etc, and derive a contradiction

Problem 5. Let $\mathcal{A}$ be a clone on $A = \{0, 1\}$ with an operation which is not essentially unary. Prove that $\mathcal{A}$ contains a constant unary operation, or at least one of the operations $\wedge, \vee, \text{maj}, \text{min}$.

Hint: try to reduce to the idempotent case

CSP lecture 24/25 – Problem Set 5

A ternary operation $m : A^3 \rightarrow A$ is called a *majority operation* if $m(a, a, b) = m(a, b, a) = m(b, a, a) = a$ for each $a, b \in A$ (note that for $|A| \leq 2$ there is a unique majority operation on A , otherwise there are more of them).

Problem 1. Let $R \subseteq A^n$ be a relation compatible with a majority operation on A . Denote $\pi_{i,j}(R)$ the projection of R onto the coordinates i, j ($1 \leq i, j \leq n$), that is,

$$\pi_{i,j}(R) = \{(a_i, a_j) : (a_1, \dots, a_n) \in R\} .$$

Prove that R is determined by these binary projections, that is,

$$(a_1, \dots, a_n) \in R \text{ if and only if } (\forall i, j, 1 \leq i, j \leq n) (a_i, a_j) \in \pi_{i,j}(R)$$

Hint: start with $n = 3$

Problem 2. Let $\mathbb{A} = (A; \dots)$ be a relational structure with a majority polymorphism. Show that there exists a relational structure $\mathbb{B} = (A; \dots)$ which contains only binary relations such that $\mathbb{A}$ is pp-definable from $\mathbb{B}$ and $\mathbb{B}$ is pp-definable from $\mathbb{A}$. For $A = \{0, 1\}$, conclude that $\text{CSP}(\mathbb{A}) \leq_P 2\text{-SAT}$ (and thus $\text{CSP}(\mathbb{A})$ is solvable in polynomial time).

Problem 2.1. Let $\mathbb{A} = (\mathbb{Z}; R_1, \dots, R_k)$, where all relations $R_1, \dots, R_k$ admit a quantifier-free definition over the relations $y < x + c$ and $y = x + c$, where $c \in \mathbb{Z}$. E.g. R can be the 4-ary relation that holds on (x, y, z, t) iff $(x > y + 1 \vee x > z - 6) \wedge (x = z \Rightarrow t = y + 1)$ holds. Suppose that the ternary median operation is a polymorphism of $\mathbb{A}$. Show that $\text{CSP}(\mathbb{A})$ is solvable in polynomial time.

Problem 3. Let $\mathbb{A} = (\{0, 1\}; \dots)$ be a relational structure with polymorphism $\min$ (from Problem Set 4). Show that each n -ary relation of $\mathbb{A}$ is an affine subspace of $\mathbb{Z}_2^n$. Conclude that $\text{CSP}(\mathbb{A})$ is solvable in polynomial time.

Problem 4. Let $\mathbb{A} = (\{0, 1\}; C_0, C_1, H)$ be as in Problem Set 1 (the corresponding CSP is HORN-3-SAT). For every relation $R \subseteq \{0, 1\}^n$ compatible with $\wedge$ find a pp-definition from $\mathbb{A}$.

Problem 5. Prove that for each relational structure $\mathbb{A} = (A; \dots)$ with $A = \{0, 1\}$, either $\text{CSP}(\mathbb{A})$ is solvable in polynomial time or $\text{CSP}(\mathbb{A})$ is NP-complete (this is *Schaefer's dichotomy theorem* (1978)). Describe the two cases in terms of polymorphisms.

CSP lecture 24/25 – Problem Set 6

An instance of $\text{CSP}(\mathbb{A})$ with set of variables V is called *1-minimal* if there exists a system of subsets $P_x \subseteq A$, $x \in V$ such that for every constraint $R(x_1, \dots, x_k)$, the projection of R onto the j -th coordinate is equal to P_{x_j} . We say the instance is *non-trivial* if none of the sets P_x is empty.

Two instances of the CSP are *equivalent* if they have the same set of solutions.

Problem 1. Devise a polynomial-time algorithm that transforms an instance of $\text{CSP}(\mathbb{A})$ to an equivalent 1-minimal instance of $\text{CSP}(\mathbb{B})$, where $\mathbb{B}$ is pp-definable in $\mathbb{A}$.

Recall that a *semilattice operation* on A is a binary operation s that is associative, commutative, and idempotent: that is, for all $a, b, c \in A$, the following equalities hold:

$$\begin{aligned} s(s(a, b), c) &= s(a, s(b, c)) \\ s(a, b) &= s(b, a) \\ s(a, a) &= a \end{aligned}$$

A *totally symmetric operation* on A of arity n is an operation $t: A^n \rightarrow A$ such that $t(a_1, \dots, a_n) = t(b_1, \dots, b_n)$ whenever $\{a_1, \dots, a_n\} = \{b_1, \dots, b_n\}$, i.e., the value of the operation only depends on the set of its arguments.

Problem 2. Give examples of semilattice operations.

Problem 2.1. Prove that every clone that contains a semilattice operation contains for every $n \geq 1$ a totally symmetric operation of arity n .

Problem 2.2. Let $\mathbb{A}$ be finite. Prove that if $\text{Pol}(\mathbb{A})$ contains totally symmetric operations of all arities $n \geq 1$, then it contains a family of totally symmetric operations $s_1, s_2, \dots$ where s_n has arity n and $s_{n+1}(x_1, x_1, x_2, \dots, x_n) = s_n(x_1, \dots, x_n)$ holds for all $x_1, \dots, x_n \in A$.

Problem 3. Suppose that $\mathbb{A}$ is a finite relational structure that has totally symmetric polymorphisms of all arities $n \geq 1$. Show that every non-trivial 1-minimal instance of $\text{CSP}(\mathbb{A})$ has a solution. Conclude that $\text{CSP}(\mathbb{A})$ is solvable in polynomial time.

Hint: apply the totally symmetric polymorphisms to the non-empty sets P_x whose existence is guaranteed by 1-minimality.

Problem 4 (Not correct). Show the converse: let $\mathbb{A}$ be finite and suppose that every non-trivial 1-minimal instance of $\mathbb{A}$ has a solution. Prove that $\text{Pol}(\mathbb{A})$ contains totally symmetric polymorphisms of all arities $n \geq 1$.

Hint: Build a non-trivial 1-minimal instance of $\text{CSP}(\mathbb{A})$ whose variables are non-empty subsets of $\mathbb{A}$, and whose solutions define totally symmetric polymorphisms of $\mathbb{A}$.

An instance of a CSP with variables $V = \{x_1, \dots, x_n\}$ over the set A is called *simple (2, 3)-minimal* if it satisfies all the following conditions:

- For each $1 \leq i \leq n$, there is a single unary constraint $P_i(x_i)$ where $P_i \subseteq A$,
- For each pair $i, j \in \{1, \dots, n\}$ of distinct integers, there is a single binary constraint $P_{i,j}(x_i, x_j)$, where $P_{i,j} \subseteq A^2$,
- $P_{i,j} = P_{j,i}^{-1}$ (i.e., $P_{i,j} = \{(b, a) \mid (a, b) \in P_{j,i}\}$),
- There are no other constraints except the ones mentioned above,
- The instance is 1-minimal: for all i, j , the restriction of $P_{i,j}$ to its first coordinate equals P_i ,
- For each triple $i, j, k \in \{1, \dots, n\}$ of distinct integers and each $(a, b) \in P_{i,j}$, there exists a $c \in P_k$ such that $(a, c) \in P_{i,k}$ and $(b, c) \in P_{j,k}$.

Problem 5. Let us represent a simple $(2, 3)$ -minimal instance as a multipartite graph as follows: each variable x_i corresponds to one set whose vertices are the elements of P_i , and for every distinct i, j and $(a, b) \in P_{i,j}$, there is an edge between the corresponding vertices $a \in P_i$ and $b \in P_j$. Describe what the last two items in the definition of $(2, 3)$ -minimality mean for this graph.

Problem 6. Let $\mathbb{A}$ be a finite structure and have only unary and binary relations. Devise a polynomial-time algorithm that transforms any instance of $\text{CSP}(\mathbb{A})$ into an equivalent simple $(2, 3)$ -minimal instance of $\text{CSP}(\mathbb{B})$, where $\mathbb{B}$ is pp-definable in $\mathbb{A}$.

Problem 7. Adapt the algorithm from the previous problem for the case where $\mathbb{A}$ has relations of arbitrary arity but $\text{Pol}(\mathbb{A})$ contains a majority operation.

Problem 8. Suppose that $\mathbb{A}$ has a majority polymorphism. Show that every non-trivial simple $(2, 3)$ -minimal instance of $\text{CSP}(\mathbb{A})$ has a solution.

Hint: if $V = \{x_1, \dots, x_n\}$ is the set of variables and $h: \{x_1, \dots, x_i\} \rightarrow A$ is an assignment that satisfies all constraints involving only the variables from $\{x_1, \dots, x_i\}$, show that h can be extended to an assignment $h': \{x_1, \dots, x_i, x_{i+1}\} \rightarrow A$ that satisfies all the constraints involving only the variables from $\{x_1, \dots, x_i, x_{i+1}\}$. Conclude that $\text{CSP}(\mathbb{A})$ is solvable in polynomial time.

Remark 1. It is also possible to characterize the property “Every non-trivial $(2, 3)$ -minimal instance of $\text{CSP}(\mathbb{A})$ has a solution” in terms of $\text{Pol}(\mathbb{A})$, although the proof is beyond the scope of the course: the property is equivalent to $\text{Pol}(\mathbb{A})$ containing for all $n \geq 3$ an operation w of arity n that satisfies

$$w(x, y, \dots, y) = w(y, x, y, \dots, y) = \dots = w(y, \dots, y, x).$$

CSP lecture 24/25 – Problem Set 7

We assume throughout the sheet that every set is finite. A *Maltsev operation* is an operation $m: A^3 \rightarrow A$ that satisfies $m(a, b, b) = m(b, b, a) = a$ for all $a, b \in A$.

Problem 1. A relation $R \subseteq A^n$ is *rectangular* if for all $i \in \{1, \dots, n\}$, all $\mathbf{a}, \mathbf{b} \in A^n$, $c, d \in A$, whenever $(a_1, \dots, a_{i-1}, c, a_{i+1}, \dots, a_n), (b_1, \dots, b_{i-1}, c, b_{i+1}, \dots, b_n), (b_1, \dots, b_{i-1}, d, b_{i+1}, \dots, b_n) \in R$, then $(a_1, \dots, a_{i-1}, d, a_{i+1}, \dots, a_n) \in R$. Show that every relation that is invariant under a Maltsev operation is rectangular.

We say that $t, t' \in A^n$ *witness* $(i, a, b) \in \{1, \dots, n\} \times A^2$ if $(t_1, \dots, t_{i-1}) = (t'_1, \dots, t'_{i-1})$ and $t_i = a, t'_i = b$. Let $R \subseteq A^n$. The *signature* of R is the set

$$\text{Sig}_R := \{(i, a, b) \in [n] \times A^2 \mid \exists t, t' \in R \text{ that witness } (i, a, b)\}.$$

We say that $R' \subseteq R$ is a *representation* of R if $\text{Sig}_{R'} = \text{Sig}_R$, and that the representation is *compact* if $|R'| \leq 2 \cdot |\text{Sig}_R|$.

Problem 2. Observe that every R has a compact representation. Describe a concrete compact representation of A^n .

Given a subset $R \subseteq A^n$ and an operation $f: A^m \rightarrow A$, the *relation generated by R under f* , denoted by $\langle R \rangle_f$, is the smallest relation S containing R and that is invariant under f . For $i_1, \dots, i_m \in \{1, \dots, n\}$, let $\pi_{i_1, \dots, i_m}(R) := \{(a_{i_1}, \dots, a_{i_m}) \mid (a_1, \dots, a_n) \in R\}$.

Problem 3. Suppose that R is invariant under a Maltsev operation f and that R' is a representation of R . Show that $\langle R' \rangle_f = R$.

Hint: Show that $\pi_{1, \dots, i}(\langle R' \rangle_f) = \pi_{1, \dots, i}(R)$, for all $i \in \{1, \dots, n\}$.

For the next exercises, we fix the following:

- $R \subseteq A^n, S \subseteq A^m$ are invariant under a Maltsev operation f ,
- $R' \subseteq R$ is a compact representation of R ,
- $i_1, \dots, i_m \in \{1, \dots, n\}$,
- The relation T is defined by $\{(a_1, \dots, a_n) \in R \mid (a_{i_1}, \dots, a_{i_m}) \in S\}$.

Problem 4. Give an algorithm that takes as input $R', (i_1, \dots, i_m), S$, and decides whether T is non-empty, in which case the algorithm should also return an arbitrary tuple in T . The running time should be polynomial in $|R'|$ and n (the arity of R').

Hint: apply the Maltsev operation to R' until the projection on the coordinates $(i_1, \dots, i_m)$ stabilises.

Problem 5. Give an algorithm that takes as input R' and c , and outputs a compact representation of $R|_c := \{(a_1, \dots, a_n) \in R \mid a_1 = c\}$ in time polynomial in $|R'|$ and n .

Hint: given any $(i, a, b) \in \text{Sig}_R$, use the algorithm from Problem 4 to decide whether (i, a, b) is in $\text{Sig}_{R|_c}$.

Note that by iterating the algorithm, one can also compute a compact representation of

$$R|_{c_1, \dots, c_m} = \{(a_1, \dots, a_n) \in R \mid a_1 = c_1, \dots, a_m = c_m\}.$$

Problem 6. Show that one can compute, in time polynomial in $|R'|$ and n , a compact representation of T .

Hint: simply describe the necessary and sufficient conditions for a given $(i, a, b) \in \text{Sig}_R$ to be in Sig_T , and use the previous two algorithms to check those conditions.

Problem 7. Prove that if $\mathbb{A}$ is a finite relational structure such that $\text{Pol}(\mathbb{A})$ contains a Maltsev polymorphism, then $\text{CSP}(\mathbb{A})$ is solvable in polynomial time.