

Universal Algebra 2 - Exercises 1

February 28, 2019

1. Show that
 - (a) a semilattice $(L, \wedge)$ is Abelian iff $|L| = 1$.
 - (b) a group $(G, \cdot, {}^{-1})$ is Abelian iff $\cdot$ is commutative.
 - (c) a ring $(R, +, 0, -, \cdot)$ is Abelian iff $x \cdot y = 0$.
 - (*) What does Abelianness mean in your favorite variety? (E.g.: Is $x \cdot y = y \cdot x$ sufficient for loops, quasigroups, monoids, semigroups...?)
2. Show that $(\mathbb{Q}, +)$ is Abelian, but has no Malcev polynomial. (Hint: Show that $+$ preserves the order $\leq$ on $\mathbb{Q}$ and that Malcev operations do not preserve linear orders). Observe that every polynomial operation $f(x_1, \dots, x_k)$ of $(\mathbb{Q}, +)$ is equal to some affine combination $\sum_{i=1}^k n_i x_i + c$ with $n_i \in \mathbb{Z}$. Why is this not a counterexample to the fundamental theorem of Abelian algebras?
3. Complete the missing proof steps of the fundamental theorem of Abelian algebras. (If $\mathbf{A}$ is Abelian with Malcev polynomial $m(x, y, z)$ why is $(A, +, 0, -)$ with $x + y := m(x, 0, y)$ and $-x := m(0, x, 0)$ an abelian group? etc.)
4. Show that in the definition of Abelian algebras it is not sufficient to require the term condition only for basic operations of $\mathbf{A}$. (Hint: groups)