# Universal Algebra 2 - Exercises 2

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An algebra **A** is called *nilpotent of (degree $n$)* if for the series of congruences $(1]^0 = 1$, $(1]^1 = [1,1]$, $(1]^2 = [[1,1],1]$,... we have $(1]^n = 0$. (Example: A group is nilpotent in this sense, if and only if it is nilpotent in the group sense.)

1. Let **R** be a commutative ring, $\alpha, \beta \in \mathrm{Con}(\mathbf{R})$, and $I$ and $J$ be the ideals corresponding to $\alpha$ and $\beta$. Show that the ideal corresponding to $[\alpha, \beta]$ is $IJ$.

2. Deduce that a commutative ring **R** is nilpotent of degree $n$ if and only if **R** satisfies the identity $x_1 \cdot x_2 \cdots x_{n+1} \approx 0$.

3. Let **A** be an algebra with Maltsev term $m(x, y, z)$ and let $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ with $[\alpha, \beta] = 0$. Let $(u, v) \in \alpha$. Then prove that $u/\beta$ and $v/\beta$ have the same size (Hint: the map $x \mapsto m(x, u, v)$).
   (*) Prove that every nilpotent **A** with Maltsev term is *congruence uniform*, i.e. for a congruence $\alpha \in \mathrm{Con}(\mathbf{A})$, all of its blocks have the same size.

4. Let **A** be an algebra with Maltsev term, such that there are pairwise different $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$ with $\alpha \wedge \beta = \alpha \wedge \gamma = \beta \wedge \gamma = 0_A$ and $\alpha \vee \beta = \alpha \vee \gamma = \beta \vee \gamma = 1_A$. Show that **A** is Abelian.