

Universal Algebra 2 - Exercises 3

March 28, 2019

1. Let $\mathcal{E} = \{f(f(x)) \approx g(x)\}$ in the language consisting of two unary operation symbols f, g .
 - (a) Show that $\mathcal{E}$ is finitely terminating.
 - (b) Show that $D(\mathcal{E})$ is not convergent.
 - (c) Add an identity to $\mathcal{E}$ to make $D(\mathcal{E})$ convergent.
2. Show that for $\mathcal{E} = \{f(f(x)) \approx f(x), f(x+y) \approx x+f(y)\}$ the digraph $D(\mathcal{E})$ is convergent. How do normal forms in $D(\mathcal{E})$ look like?
3. Let $\mathcal{E}$ consist of the single identity $w(x, y, z) \approx w(y, z, x)$. Show that there is no $\mathcal{F}$ equivalent to $\mathcal{E}$ such that $D(\mathcal{F})$ is convergent. What seems to be the problem? (*) Any ideas on how to fix it?
4. Give an example of a digraph that is locally confluent, but not confluent.
5. Finish the proof of the Equational completeness theorem: Show that for any signature τ and any set of equations $\mathcal{E}$, the set

$$\{(s, t) \in F^2 \mid \mathcal{E} \vdash s \approx t\}$$

is a fully invariant congruence of the free algebra $\mathbf{F} = \mathbf{F}_\tau(x_1, x_2, \dots)$.