Universal Algebra 2 - Exercises 4

April 11, 2019

1. We show that the variety of Abelian groups $\mathcal{A}$ does not have DPC:

   (a) Verify that in every Abelian group: $(c,d) \in \mathrm{Cg}(a,b)$ iff there is a $n \in \mathbb{Z}$ such that $(c-d) = n \cdot (a-b)$

   (b) Prove that, if $\mathcal{A}$ had DPC there would be a $k \in \mathbb{Z}$ such that the formula

   $$\bigvee_{|n| \leq k} (c-d) = n \cdot (a-b)$$

   is equivalent to $(c,d) \in \mathrm{Cg}(a,b)$ in $\mathcal{A}$. (Hint: compactness argument from lecture)

   (c) Show that (b) does not hold in $\mathbb{Z}$.

2. Prove that the variety of semilattices has DPC. (Hint: there are only very few unary polynomials in a semilattice.)

3. Let $\mathbf{L}$ be a distributive lattice. Prove that $(c,d) \in \mathrm{Cg}^{\mathbf{L}}(a,b)$ iff

   $$c \wedge (a \wedge b) = d \wedge (a \wedge b) \,\&\, c \vee (a \vee b) = d \vee (a \vee b).$$

   Thus the variety of distributive lattices has DPC. Conclude (with the help of Jónsson's lemma) that every distributive lattice is finitely based.

4. Let $\mathcal{V}$ be a locally finite variety and suppose that ψ is a congruence formula that defines principal congruences of all finite elements of $\mathcal{V}$. Show that ψ defines principal congruences on all elements of $\mathcal{V}$.