

Universal Algebra 2 - Exercises 6

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1. Let $\mathbf{G} = (G, x - y + z)$ be a commutative group and $\mathbf{H}$ be a proper subgroup. Prove that $\mathbf{H}$ does not absorb $\mathbf{G}$.
2. ('Walking'): Let $\mathbf{A}$ be an algebra and $R \leq_{sd} \mathbf{A}^2$. Let $\mathbb{P}$ be an oriented path and let $B \subseteq A$. Denote by $B^{+\mathbb{P}}$ the set of $a \in A$ for which there exists a $b \in B$ from which there is a $\mathbb{P}$ -shaped path from b to a (using edges from R). (E.g. $B^{+\rightarrow}$ is the set of all out-neighbors of B). Prove that:
 - $B \leq \mathbf{A} \Rightarrow B^{+\mathbb{P}} \leq \mathbf{A}$
 - $B \trianglelefteq_t \mathbf{A} \Rightarrow B^{+\mathbb{P}} \trianglelefteq_t \mathbf{A}$
3. Show that in every finite idempotent algebra $\mathbf{A}$, there is a term s such that for every $B \subseteq A$ we have $s(B, \dots, B) = \text{Sg}^{\mathbf{A}}(B)$ (completing the proof of the existence of transitive terms).
4. Prove that the loop lemma (Theorem 1) implies that every finite, idempotent Taylor algebra $\mathbf{A}$ has a *Siggers operation*: a 4-ary operation s satisfying the identity $s(x, z, x, y) \approx s(z, x, y, z)$. (Hint: Apply the loop lemma to the free algebra $\mathbf{F} = \mathbf{F}_{\text{HSP}(\mathbf{A})}(x, y, z)$; and let $R \leq \mathbf{F}^2$ be the subalgebra generated by the pairs $(x, z), (z, x), (x, y), (y, z)$)
5. Let $\mathbb{G} = (A; R)$ be a digraph. The smooth part of $\mathbb{G}$, which we denote by $\text{smooth}(\mathbb{G})$, is the largest subset of $U \subseteq A$ such that the digraph $(U, R \cap (U \times U))$ is smooth. Find a primitive positive definition for $\text{smooth}(\mathbb{G})$ (the definition might depend on G).

Theorem 1 (Loop lemma). *Let $\mathbf{A}$ be a finite, idempotent Taylor algebra. Let $\mathbb{G} = (A; R)$ be a smooth digraph of algebraic length 1 such that $R \leq \mathbf{A} \times \mathbf{A}$. Then R has a loop, i.e. $\exists a \in A : R(a, a)$*