Convex Optimization

Homework # 1

Instructions

- The solutions are expected to be written in LaTeX and saved in the `.pdf` format.
- Please, submit your homeworks to alexey.barsukov@matfyz.cuni.cz. The subject of your email should be of the following form: `Convex Optimization HW *#ofHW* *yourSurname* *yourName*`, for example: `Convex Optimization HW 1 Hašek Jaroslav`
- There will be 4 homework assignments, on each of which you need to score at least 60 out of 100 points to obtain the credit (zápočet) for the course.
- Please, send your submissions not later than October, 30, 23:59 local time.
- You can (and are encouraged to) work in groups, but each participant must write their own solution separately. For numerical problems, you may want to use CVXPY.

Exercise 1 (15 points)

Sketch the set given by the following set of inequalities for the variables x_1 and x_2:

$$\begin{aligned} x_1 &\geq 0 \\ x_1 + x_2 &\geq -1 \\ -3x_1 + x_2 &\leq 4 \end{aligned}$$

Is this set open or closed (in the standard Euclidian topology on $\mathbb{R}^2$)? Is this set convex? Is this set bounded? Describe the boundary of the given set in terms of linear equalities and inequalities.

Exercise 2 (20 points)

Consider the usual linear regression problem, i.e., given matrix A and vector $\mathbf{b}$, our goal is to find $\mathbf{x}$ minimizing $\|A\mathbf{x} - \mathbf{b}\|_2^2$. Show that this problem can be solved by solving the system of normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$. You may use without the proof a fact that the x minimizing the above objective can be found by solving the equation

$$\nabla_{\mathbf{x}}\|A\mathbf{x} - \mathbf{b}\|_2^2 = 0$$

where $\nabla_{\mathbf{x}} F$ denotes the gradient of F (although different arguments can be used to derive the normal equations). Show that the given system is solvable even if $A^T A$ is singular.

Exercise 3 (15 points)

Let $S_1 = \{\mathbf{x} \in \mathbb{R}^4 \mid \|\mathbf{x}\|_2 \leq 2\}$, and $S_2 = \{\mathbf{x} \in \mathbb{R}^4 \mid \|\mathbf{x} - \mathbf{c}\|_2 \leq 3\}$, where $\mathbf{c}^T = (-1,\ -2,\ 3,\ 4)$. Argue why S_1 and S_2 are both convex and disjoint and find a hyperplane that strictly separates them.

Exercise 4 (20 points)

1. Prove that the function $f(x_1, x_2) = \log(e^{x_1} + e^{x_2})$ is a convex function on $\mathbb{R}^2$. Find a convex function $g\colon \mathbb{R}^2 \to \mathbb{R}$ so that $\log(g)$ is not convex.
2. Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}$ is nonnegative and convex, and $g\colon \mathbb{R}^n \to \mathbb{R}$ is positive and concave. Show that the function f^2/g, with domain $\mathrm{dom}(f) \cap \mathrm{dom}(g)$, is convex.

Exercise 5 (30 points)

- A company makes three products (X, Y and Z) using two available machines (A and B).
- To produce a unit of X one requires 45 minutes of processing time on machine A and 30 minutes of processing time on machine B. For Y, one requires 20 minutes on A and 10 minutes on B and, for Z, one requires 40 minutes on A and 25 minutes on B.
- At the start of the current week, there are 30 units of X, 100 units of Y, and 40 units of Z in stock.
- Due to technical limitations one can use the machine A for no more than 150 hours and the machine B for no more than 120 hours through the week.
- The demand for X in the current week is 75 units, for Y, it is 180 units, and, for Z, it is 45 units. Company policy is to meet the demand by the end of the week and to maximize the weighted sum of number of units X, Y, and Z left in stock after the demanded products are sold. The weight of each product type equals its demand in the current week (so we are trying to accumulate more products for which the demand is higher).

Formulate this problem as a linear program and solve it using software. To get the full points you need to describe the linear program and how did you get it in written form. You do not need to send a source code for the program solving the problem.