Convex Optimization

Homework # 3

Instructions

Please, submit your homeworks to alexey.barsukov@matfyz.cuni.cz not later than December, 4, 23:59 local time. The subject of your email should be of the following form: `Convex Optimization HW *#ofHW* *yourSurname* *yourName*`, for example: `Convex Optimization HW 3 Einstein Albert`

Exercise 1 (10 points)

What is the solution of the norm approximation problem with one scalar variable $x \in \mathbb{R}$:

$$\text{minimize} \qquad \|x\mathbf{1} - b\|$$

for ℓ_1-, ℓ_2-, and ℓ_∞-norms? Here, $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ are constant vectors.

Exercise 2 (15 points)

The problem

$$\begin{array}{ll} \text{minimize} & -3x_1^2 + x_2^2 + 2x_3^2 + 2(x_1 + x_2 + x_3) \\ \text{subject to} & x_1^2 + x_2^2 + x_3^2 = 1 \end{array}$$

is an example of a non-convex problem for which strong duality holds. Derive the KKT conditions. Find all solutions x, ν that satisfy the KKT conditions. Which pair corresponds to the optimum?

Exercise 3 (25 points)

Consider the quadratic program

$$\begin{array}{ll} \text{minimize} & x_1^2 + 2x_2^2 - x_1x_2 - x_1 \\ \text{subject to} & x_1 + 2x_2 \le u_1 \\ & x_1 - 4x_2 \le u_2 \\ & 5x_1 + 76x_2 \le 1 \end{array}$$

with variables x_1, x_2, and parameters u_1, u_2.

(a) Solve this QP (using `cvxpy`), for parameter values $u_1 = -2$, $u_2 = -3$, to find optimal primal variable values $x_1^\star$ and $x_2^\star$, and optimal dual variable values $\lambda_1^\star$, $\lambda_2^\star$, and $\lambda_3^\star$. Let $p^\star$ denote the optimal objective value. Verify that the KKT conditions hold for the optimal primal and dual variables you found (within reasonable numerical accuracy).

(b) We will now solve some perturbed versions of the QP, with

$$u_1 = -2 + \delta_1, \qquad u_2 = -3 + \delta_2,$$

where δ_1 and δ_2 each take values from $\{-0.1, 0, 0.1\}$. (There are a total of nine such combinations, including the original problem with $\delta_1 = \delta_2 = 0$.) For each combination of δ_1 and δ_2, make a prediction $p^\star_{\text{pred}}$ of the optimal value of the perturbed QP, and compare it to $p^\star_{\text{exact}}$, the exact optimal value of the perturbed QP (obtained by solving the perturbed QP). Put your results in the two righthand columns in a table with the form shown below. Check that the inequality $p^\star_{\text{pred}} \le p^\star_{\text{exact}}$ holds.

δ_1	δ_2	$p^\star_{\text{pred}}$	$p^\star_{\text{exact}}$
0	0		
0	-0.1		
0	0.1		
-0.1	0		
-0.1	-0.1		
-0.1	0.1		
0.1	0		
0.1	-0.1		
0.1	0.1		

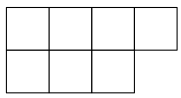
Exercise 4 (25 points)

There are two players, P and Q. Each player chooses a number from $\{1, 2, 3\}$ that describe a kind of attack/defense. If P's number is equal to Q's then Q has managed to deflect P's attack and both players gain 0 points. If the numbers differ then P gains and Q loses the number of points equal to the number chosen by P.

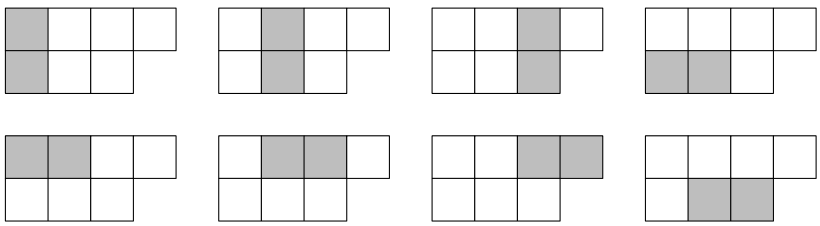
Determine a worst case optimal strategy for P and Q. Calculate the value of this game (that is, expected number of points gained by P assuming both players play their worst case optimal strategies). You do not have to send your code, but explain what your optimization problem was and how it connects to the game.

Exercise 5 (25 points)

Consider the following version of the game of Battleship: There is just one round. Player 1 places a 1×2 ship somewhere in the irregular playing field which looks like in the picture



with 8 possible ship positions:



Player 2, not knowing Player 1's choice, picks one of the 7 squares to shoot at. If Player 2's shot hits the ship, Player 1 loses a point and Player 2 gains a point, otherwise Player 1 gains a point and Player 2 loses a point.

Use `cvxpy` to calculate the worst-case optimal strategy for both players and the value of this game. You do not have to send your code, but explain what your optimization problem was and how it connects to the game.