Convex Optimization

Practical session # 5

October 30, 2024

Exercise 1 (Ex. 5 of the last week). Formulate the ℓ_4-norm approximation problem

$$\text{minimize} \qquad \|A\mathbf{x} - \mathbf{b}\|_4$$

as a QCQP. The matrix $A \in \mathbb{R}^{m \times n}$ and the vector $\mathbf{b} \in \mathbb{R}^m$ are fixed.

Exercise 2. Let K_{pol} be the set of (coefficients of) nonnegative polynomials of degree $2k$ on $\mathbb{R}$:

$$K_{\text{pol}} = \{x \in \mathbb{R}^{2k+1} \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_{2k+1} t^{2k} \geq 0 \text{ for all } t \in \mathbb{R}\}.$$

1. Show that K_{pol} is a proper cone.
2. A basic result states that a polynomial of degree $2k$ is nonnegative on $\mathbb{R}$ if and only if it can be expressed as the sum of squares of two polynomials of degree k or less. In other words, $x \in K_{\text{pol}}$ if and only if the polynomial
$$p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_{2k+1} t^{2k}$$
can be expressed as
$$p(t) = r(t)^2 + s(t)^2,$$
where r and s are polynomials of degree k. Use this result to show that
$$K_{\text{pol}} = \left\{ x \in \mathbb{R}^{2k+1} \mid x_i = \sum_{m+n=i+1} Y_{mn} \text{ for some } Y \in \mathbb{S}_+^{k+1} \right\}$$
In other words, $p(t)$ is nonnegative if and only if there exists a matrix $Y \in \mathbb{S}_+^{k+1}$ such that
$$\begin{aligned} x_1 &= Y_{11} \\ x_2 &= Y_{12} + Y_{21} \\ x_3 &= Y_{13} + Y_{22} + Y_{31} \\ &\vdots \\ x_{2k+1} &= Y_{k+1,k+1} \end{aligned}$$

Exercise 3. Recall that *semidefinite program* (SDP) has the form

$$\begin{aligned} \text{minimize} \quad & c^T x, \\ \text{subject to} \quad & x_1 F_1 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b, \end{aligned}$$

where $G, F_1, \ldots, F_n \in \mathbb{S}^k$ are symmetric matrices, and $A \in \mathbb{R}^{p \times n}$.

Let $p(t)$ be a polynomial of the same form as in Exercise 2. Consider an optimization problem, where the goal is to find such a polynomial which has the greatest minimal value and which satisfies the bounds $\ell_i \leq p(t_i) \leq r_i$ at m fixed points t_i:

$$\begin{aligned} \text{maximize} \quad & \inf_t p(t) \\ \text{subject to} \quad & \ell_i \leq p(t_i) \leq r_i, \quad i = 1, \ldots, m, \end{aligned}$$

where the variables are the coefficients $x_1, \ldots, x_{2k+1} \in \mathbb{R}$. Find an equivalent SDP problem.

Hint: Use the conditions of $p(t)$ being nonnegative that you obtained in Exercise 2.

Exercise 4. Let $X \in \mathbb{S}^n$ be a symmetric matrix of the form

$$X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

where $A \succ 0$ is positive definite. Let $S = C - B^T A^{-1} B$ be the *Schur complement* of A. Show that $X \succeq 0$ if and only if $S \succeq 0$.

Hint: Find a matrix U such that

$$U^T X U = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}$$

Exercise 5. Use the result of Exercise 4 to formulate the QP, the QCQP, and the SOCP as SDPs.