Convex Optimization

Practical session # 8

November 20, 2024

Exercise 1

Consider the QCQP

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & (x_1 - 1)^2 + (x_2 - 1)^2 \le 1 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \le 1 \end{array}$$

with variable $x \in \mathbb{R}^2$.

(a) Sketch the feasible set and level sets of the objective. Find the optimal point $x^\star$ and optimal value $p^\star$.

(b) Give the KKT conditions. Do there exist Lagrange multipliers $\lambda_1^\star$ and $\lambda_2^\star$ that prove that $x^\star$ is optimal?

(c) Derive and solve the Lagrange dual problem. Does strong duality hold?

Exercise 2

Solve the following problem using KKT:

$$\begin{array}{ll} \text{minimize} & 4x + 5y + 3z \\ \text{subject to} & x^2 + 2y^2 + z^2 \le 4 \end{array}$$

Exercise 3

Let $f_0, f_1, \ldots, f_m \colon \mathbb{R}^n \to \mathbb{R}$ be convex. Show that the optimal value for the perturbed problem $p^\star(u, v)$ is convex as a function of u, v, where

$$p^\star(u,v) = \inf\big\{ f_0(x) \mid \exists x \in \mathbb{R}^n \text{ s.t. } f_i(x) \le u_i,\ i = 1, \ldots, m, \quad Ax - b = v \big\}$$

Exercise 4

Let $\phi \colon \mathbb{R} \to \mathbb{R}$ be the log barrier penalty function with limit $a > 0$:

$$\phi(x) = \begin{cases} -a^2 \log\left(1 - \left(\frac{x}{a}\right)^2\right) & |u| < a \\ \infty & \text{otherwise.} \end{cases}$$

Show that if $u \in \mathbb{R}^m$ satisfies $\|u\|_\infty < a$, then

$$\|u\|_2^2 \le \sum_{i=1}^m \phi(u_i) \le \frac{\phi(\|u\|_\infty)}{\|u\|_\infty^2} \|u\|_2^2$$

This means that $\sum_{i=1}^m \phi(u_i)$ is well approximated by $\|u\|_2^2$ if $\|u\|_\infty$ is small compared to a. For example, if $\|u\|_\infty / a = 0.25$, then

$$\|u\|_2^2 \le \sum_{i=1}^m \phi(u_i) \le 1.033 \cdot \|u\|_2^2$$