Convex Optimization

Practical session # 9

November 27, 2024

Exercise 1 Show that following optimization problems that approximately solve $Ax \approx b$ (for $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$) are equivalent to some "nice" convex problem (LP, QP, SOCP, SDP,...)

(a) *deadzone-linear penalty approximation*

minimize $\sum_{i=1}^m \phi(a_i^T x - b_i)$, for $\phi(u) = \begin{cases} 0 & \text{if } |u| < d \\ |u| - d & \text{if } |u| \geq d \end{cases}$

(b) *largest k residuals*

minimize $\sum_{i=1}^k |r|_{[i]}$; subject to $r = Ax - b$,
where $|r|_{[1]} \geq |r|_{[2]} \geq \ldots \geq |r|_{[m]}$ stand for the residuals $|r_1|, |r_2|, \ldots, |r_m|$ sorted in decreasing order.

(c) *Log-Chebyshev approximation*

minimize $\max_{i=1,\ldots,m} |\log(a_i^T x) - \log(b_i)|$ (assuming $b \succ 0$).

Exercise 2 (*Minmax rational function fitting*)

We are given some datapoints $(t_i, u_i) \in \mathbb{R}^2$ for $i = 1, \ldots, k$ with $t_i \in [\alpha, \beta]$ and want to fit a rational function $f(t) = p(t)/q(t)$ to them, where $p(t) = a_0 + a_1 t + \ldots + a_m t^m$, $q(t) = 1 + b_1 t + \ldots + b_n t^n$ (with fixed n, m and $q(t) > 0$ on $[\alpha, \beta]$). Define the corresponding $\|\cdot\|_\infty$-approximation problem and show that it is quasiconvex.

Exercise 3 (*Fitting a concave quadratic function*)

(a) We are given the datapoints $x_1, \ldots, x_N \in \mathbb{R}^n$, $y_1, \ldots, y_N \in \mathbb{R}$, and wish to find a *concave* quadratic function of the form

$$f(x) = (1/2)x^T P x + q^T x + r,$$

with $f(x_i) \approx y_i$. Describe this as a (constrained) norm approximation problem.

(b) For the $\|\cdot\|_2$-norm, show that this problem is equivalent to an SDP.

(c*) Let $B = \{x \mid l \preceq x \preceq u\}$ for some fixed $l \prec u$, and let us assume $x_i \in B$ for all i. Formulate a convex optimization problem under the additional constraints that f is *non-negative* and *increasing* on the box B (i.e. $0 \leq f(z) \leq f(z')$ for all $z, z' \in B$ with $z \preceq z'$). Try to simplify it as much as possible.

Exercise 4 *(Fitting a convex function)*

(a) Given some datapoints $x_1, \ldots, x_N \in \mathbb{R}^n$, $y_1, \ldots, y_N \in \mathbb{R}$, show that there is a convex function $f\colon \mathbb{R}^n \to \mathbb{R}$ with $f(x_i) = y_i$, if and only if there are vectors $g_1, \ldots, g_N \in \mathbb{R}^n$ with

$$y_i + g_i^T(x_j - x_i) \leq y_j \text{ for all } i, j = 1, \ldots, N.$$

(Hint: supporting hyperplanes of $epi(f)$).

(b) use (a) to construct a convex optimization problem (QP) that finds the optimal $\|\cdot\|_2$ approximation of a dataset by a convex function.