Convex Optimization 2025/26

Practical session # 12

January 8, 2025

1. A convex function $f\colon \mathbf{R} \to \mathbf{R}$ is called *self-concordant* if $|f'''(x)| \leq 2f''(x)^{3/2}$, for all $x \in \mathrm{dom}(f)$.

   A general function $f\colon \mathbf{R}^n \to \mathbf{R}$ is self-concordant, if $t \mapsto f(x+tu)$ is self concordant for all $x \in \mathrm{dom}(f)$ and $u \in \mathbf{R}^n$. Show that the following are self-concordant:

   - affine functions
   - quadratic functions
   - $f(x) = -\log(x)$
   - $f(x) = x\log(x) - \log(x)$

2. Show that self-concordant functions are closed under

   - sums, scaling with $\alpha \geq 1$
   - precomposition with affine functions

3. Use the above to prove that the objective functions that we get for computing the central path $x^*(t)$ of a linear program is self-concordant.

4. Let $x^{(0)}, x^{(1)}, \dots$ be the values computed by Newton descent for minimizing $f(x)$. If f is self-concordant, then we know there are numbers $\gamma > 0$ and $0 < \eta \leq 1/4$ such that

   - (damped phase) If $\lambda(x^{(k)}) > \eta$, then $f(x^{(k+1)}) - f(x^{(k)}) \leq -\gamma$,
   - (quadratically convergent phase) If $\lambda(x^{(k)}) \leq \eta$, then $2\lambda(x^{(k+1)}) \leq (2\lambda(x^{(k)}))^2$.

   The main advantage of self-concordance is that we can set the constants to $\eta = \frac{1-2\alpha}{4}$ and $\gamma = \alpha\beta\frac{\eta^2}{1+\eta}$, which do not depend on f, but only the backtracking parameters α, β.

   Using the above inequalities show that we need at most

   $$\frac{f(x^{(0)}) - p^*}{\gamma} + \log_2\log_2(\epsilon^{-1}) = \frac{20-8\alpha}{\alpha\beta(1-2\alpha)^2}(f(x^{(0)}) - p^*) + \log_2\log_2(\epsilon^{-1})$$

   Newton iterations to reach tolerance $\epsilon > 0$. (so, for values $\alpha = 0.1, \beta = 0.8$ and $\epsilon = 10^{-10}$, we get e.g. $375(f(x^{(0)}) - p^*) + 6$).

5. Let us consider one centering step of the log-barrier method.

   - (*) If $x = x(t)$ and $x^+ = x(\mu t)$, then one can prove that

     $$\mu t f_0(x) + \phi(x) - \mu t f_0(x^+) - \phi(x^+) \leq m(\mu - 1 - \log\mu)$$

   - Use (3), (4) and (*) to come up with an upper bound of the total number of Newton iterations that is used in solving a linear program using the log-barrier method.
   - Try to prove (*) (Hint: Use that $\lambda_i^*(t) = -1/(tf_i(x))$, and $\log(u) \leq u - 1$)