Multiple Objectives

Very often, plausible economic decisions cannot be chosen only according to one criterion, such as the maximal profit, production efficiency or yield. In production planning, environmental criteria have to be taken into account, in macroeconomical problems regional aspects such as the local unemployment level play an essential role. Very different, even conflicting goals can be set for a short time horizon and for the long one, etc. The mentioned disparate criteria will hardly be satisfied by a uniformly optimal decision.

Investment decisions should hedge against risks of various kinds, such as liquidity, volatility or currency risks.

Problems of the above kind belong under *multi-objective programming*. We shall briefly introduce main approaches for the case of continuous decision variables and illustrate them in the context of portfolio optimization and risk management.

The problem:

"minimize" $K \geq 2$ functions $f_1, \ldots, f_K$, $f_k : R^n \to R^1$ on a closed set $\mathcal{X}$

briefly

$$\text{"min" } \mathbf{f}(\mathbf{x}) \text{ on } \mathcal{X}.$$

Ideal solution $\tilde{\mathbf{x}} \in \mathcal{X}$ of the multi-objective programming problem:

$$\tilde{\mathbf{x}} \in \bigcap_{k=1}^{K} \arg\min_{\mathbf{x} \in \mathcal{X}} f_k(\mathbf{x}). \tag{1}$$

Ideal solutions exist only rarely.

Definition. Solution $\hat{\mathbf{x}} \in \mathcal{X}$ is an **efficient solution** of multi-objective problem if there is no element $\mathbf{x} \in \mathcal{X}$ for which $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\hat{\mathbf{x}})$ and $\mathbf{f}(\mathbf{x}) \neq \mathbf{f}(\hat{\mathbf{x}})$.

How to get efficient solutions?

Theorem. Let $\mathcal{X}$ be compact, $f_k, k = 1, \ldots, K$, continuous on $\mathcal{X}$ and $h : R^K \to R^1$ arbitrary continuous function nondecreasing in its arguments. Then at least one solution belonging to

$$\mathcal{X}_h^* := \arg\min_{\mathbf{x}\in\mathcal{X}} h(f_1(\mathbf{x}), \ldots, f_K(\mathbf{x}))$$

is efficient for the multi-objective problem (1).

Comment

Optimal solution and numerical tractability of the optimization problem depend substantially on the choice of function h.

It would not be necessary to deal with multi-objective programming if the choice of h was straightforward.

Special simple choice of function h

$$h(\mathbf{z}) = \sum_{k=1}^{K} t_k z_k \text{ with vector parameter } \mathbf{t} \in R_+^K,\ \mathbf{t} \neq 0.$$

Using possible choices of weights $\mathbf{t}$, one may find *efficient solutions* of the solved multi-objective problem:

Theorem. For vector parameter $\mathbf{t} \in R_+^K$, let $\bar{\mathbf{x}}$ be an optimal solution of

$$\min_{\mathbf{x} \in \mathcal{X}} \sum_{k=1}^{K} t_k f_k(\mathbf{x}). \qquad (2)$$

Assume that parameter vector $\mathbf{t}$ is positive OR that $\bar{\mathbf{x}}$ is unique optimal solution of (2). Then $\bar{\mathbf{x}}$ is efficient solution of the multi-objective program (1).

Convexity

For *convex* f_k, $\forall k$ and for $\mathcal{X}$ nonempty, convex, compact $\Longrightarrow$ for arbitrary efficient solution $\bar{\mathbf{x}}$ of (1) there exists $\mathbf{t} \in R_+^K$ so that $\bar{\mathbf{x}}$ is optimal solution of (2).

Stronger result holds true for *linear* functions f_k, $\forall k$ and a convex polyhedral set $\mathcal{X}$:

$\hat{\mathbf{x}}$ is an efficient solution of the multi-objective problem (1) $\Longleftrightarrow$ it is an optimal solution of (2) for a *positive* parameter vector $\mathbf{t} \in R_+^K$.

$\exists$ other approaches that provide efficient solutions

ϵ-Constrained Approach

Select one of considered objective functions, say, f_1, choose a threshold vector $\epsilon \in R^{K-1}$ and solve the classical optimization problem

$$\text{minimize } f_1(\mathbf{x}) \text{ subject to } \mathbf{x} \in \mathcal{X} \text{ and } f_k(\mathbf{x}) \leq \epsilon_k,\ k = 2, \ldots, K. \qquad (3)$$

If the set $\mathcal{X}_\epsilon := \{\mathbf{x} \in \mathcal{X} | f_k(\mathbf{x}) \leq \epsilon_k,\ k = 2, \ldots, K\} \neq \emptyset$, then

(i) the unique optimal solution $\bar{\mathbf{x}}$ of (3) is an efficient solution of (1).

(ii) Let $\hat{\mathbf{x}}$ be an efficient solution of (1). Then there exists $\epsilon \in R^{K-1}$ such that $\hat{\mathbf{x}}$ is an optimal solution of (3).

Mixed Approach is related to a different treatment of distinct objective functions: some of them are put into constraints as in the ϵ-constrained approach whereas weights $t_k > 0$ are assigned to the remaining objective functions.

Goal Programming

The main idea: Get solution from $\mathcal{X}$ for which the outcome measured by vector $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_K(\mathbf{x}))^\top$ is as close as possible to the K-vector of the best attainable outcomes $f_k^* = \min_{\mathbf{x}\in\mathcal{X}} f_k(\mathbf{x}), k = 1, \ldots, K$.

The distances are defined in the space of *function values*, a subset of R^K, and one is free to use any of (weighted) L_p-distances, $1 \leq p \leq \infty$. Hence, one solves a minimization problem

$$\min_{\mathbf{x}\in\mathcal{X}} \|\mathbf{T}(\mathbf{f}^* - \mathbf{f}(\mathbf{x}))\|_p \tag{4}$$

with a diagonal matrix $\mathbf{T} = \mathrm{diag}\{t_1, \ldots, t_K\}$, $t_k > 0\,\forall k$.

The following statements are easy to prove:

(i) Let $\bar{\mathbf{x}}$ be an optimal solution of (4) with $1 \leq p < \infty$. Then $\bar{\mathbf{x}}$ is an efficient solution of (1).

(ii) For $p = \infty$, at least one of optimal solutions of the minimax problem

$$\min_{\mathbf{x}\in\mathcal{X}} \max_k t_k |f_k^* - f_k(\mathbf{x})|$$

is efficient for (1).

Conclusions

Techniques of multi-objective programming allow to exclude "bad", non-efficient solutions of the multi-objective problems (1) provided that the selected criteria can be quantified and that the considered objective functions are a priori given the same importance.

In interactive numerical approaches the user is allowed to change the values of parameters (weights and thresholds) to achieve an acceptable balance between the criteria. There are various other problems, e.g., an appropriate treatment of hierarchically ordered objective functions or the case of a finite list of feasible alternatives.

Applications in statistics, finance and in modeling stochastic decision problems.