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STRESS TESTING FOR RISK-AVERSE STOCHASTIC PROGRAMS

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ABSTRACT. Possible use of the contamination technique in stress testing of risk measures and risk-averse stochastic programs was initiated by Dupačová and Polívka [**9**] and detailed for the Value at Risk (VaR) and the Conditional Value at Risk (CVaR). In this paper we discuss several extensions of the approach, namely to stress testing for multistage risk-averse stochastic programs with CVaR related objectives, and for spectral and polyhedral risk measures.

1. INTRODUCTION

Classical stochastic programming models aim at hedging against consequences of possible realizations of random parameters so that the final expected outcome or position is the best possible. However, such formulations do not capture the risk which is an important issue in finance, natural resources management and other fields. We shall deal with stochastic programs that can be put into the following form:

$$\text{Minimize } F(x, P) \text{ with respect to } x \in \mathcal{X} \subset \mathbb{R}^n \tag{1}$$

with P the probability distribution of the random parameters $\omega \in \Omega$ that enter the problem formulation, F concave in P and $\mathcal{X}$ a closed, nonempty set that *does not depend on* P. After reformulation to the form (1), the main decision variable x is typically the first-stage decision.

In the context of risk management, problems with $F(x, \bullet)$ *linear* in P correspond to risk neutral objectives or to minimization of the negative expected utility of the random outcome z of decision $x \in \mathcal{X}$. The risk-averse objective function F in (1) is based on a risk measure. It assigns a real number to each random outcome z from a certain class $\mathcal{Z}$, a linear space of allowable outcomes, defined on the probability space $(\Omega, \mathcal{B}, P)$. It depends only on the probability distribution of the random outcome of decision x – a property of law invariant (version-independent) risk measures. In addition, several properties of risk measures, e.g. coherence (monotonicity, translation equivariance, convexity, positive homogeneity, cf. Artzner et

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al. [**2**]), are mostly assumed. The popular examples of risk measures involve variance, Value at Risk (VaR) and Conditional Value at Risk (CVaR). The first two are not coherent, while the latter is.

In program (1), P plays a role of an abstract parameter. Accordingly, the optimal value of (1) will be denoted $\varphi(P)$ and the set of optimal solutions $\mathcal{X}^*(P)$. In applications, complete knowledge of P is rare and it is usually estimated based on past information and/or an expert opinion. Therefore it is necessary to apply a suitable output analysis to draw conclusions about stability or robustness of the obtained results with respect to changes of P. There are various statements about persistence, stability and sensitivity for parametric programs of the above type, see e.g. Römisch [**21**] and references therein. In this paper we embed the problem (1) into a family of optimization problems parametrized by a **scalar** parameter t. This family results from contamination of the original probability distribution P by another **fixed** probability distribution Q, i.e., from using distributions P_t of the form

$$P_t = (1-t)P + tQ \qquad \text{with } t \in [0,1] \tag{2}$$

in the objective function of (1) at the place of P. This parametrization does not require any specific properties of P and leads to applicable results related to stress testing of the portfolio performance.

We begin with a brief summary of the contamination technique (cf. Dupačová [**4, 5**]) for the general form of stochastic programs (1), with the expectation-type of the objective function as a special instance. Results for the CVaR objective function are given in Section 3 and are extended to a multiperiod problem. Section 4 is devoted to an extension to spectral risk measures with a discussion related to a possible exploitation for polyhedral risk measures and general coherent risk measures.

2. Contamination technique

For fixed probability distributions P, Q, let

$$F_Q(x,t) = F(x,(1-t)P + tQ)$$

denote the contaminated objective function. For fixed x, it is evidently a concave function on $[0,1]$. We denote further

$$\varphi_Q(t) = \inf_{x\in\mathcal{X}} F_Q(x,t) \qquad \text{and} \qquad \mathcal{X}_Q^*(t) = \arg\min_{x\in\mathcal{X}} F_Q(x,t) \tag{3}$$

the optimal value function and the set of optimal solutions of the perturbed stochastic program

$$\text{minimize } F_Q(x,t) \text{ with respect to } x \in \mathcal{X}. \tag{4}$$

There are various statements concerning properties of parametric program (4) with a scalar parameter t; see e.g. Gol'shtein, [**13**, Chapter 7] or Bonnans and Shapiro, [**3**, Section 4.3]:
Under the additional assumption that the set $\mathcal{X}^*(P) = \mathcal{X}_Q^*(0)$ of optimal solutions of the original problem (1) is nonempty and bounded and that $\mathcal{X}_Q^*(1) \neq \emptyset$, the

function φ_Q is continuous at $t = 0$ (cf. Gol'shtein [**13**, Theorem 15]) and its value at $t = 0$ equals the optimal value of (1): $\varphi_Q(0) = \varphi(P)$.

Under modest additional assumptions, the directional derivative of the optimal value $\varphi_Q(t)$ of the perturbed program (4) at $t = 0$ equals

$$\varphi'_Q(0^+) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_Q(0^+) = \min_{x\in\mathcal{X}^*(P)} \frac{\mathrm{d}}{\mathrm{d}t}F_Q(x, 0^+). \tag{5}$$

The form of contamination bounds for $\varphi_Q(t)$ is then a consequence of its concavity

$$(1-t)\varphi(P) + t\varphi(Q) \le \varphi_Q(t) \le \varphi(P) + t\varphi'_Q(0^+) \quad \text{for all } t \in [0,1]\,. \tag{6}$$

In the risk-neutral case with $F(x, P)$ *linear* in P, $F_Q(x, t)$ is a linear function of t and for an arbitrary fixed x, (5) reduces to

$$\varphi'_Q(0^+) = \min_{x\in\mathcal{X}^*(P)} [F(x, Q) - F(x, P)] = \min_{x\in\mathcal{X}^*(P)} F(x, Q) - \varphi(P). \tag{7}$$

Contamination bounds (6) can be then relaxed to

$$(1-t)\varphi(P) + t\varphi(Q) \le \varphi(P_t) \le (1-t)\varphi(P) + tF(x(P), Q) \tag{8}$$

valid for an arbitrary $x(P) \in \mathcal{X}^*(P)$ and $t \in [0, 1]$.

Concavity of the optimal value function $\varphi_Q(t)$ cannot be obtained, in general, when the set $\mathcal{X}$ depends on the probability distribution P or when F is not concave in P. In such cases and under additional assumptions, only local contamination bounds can be constructed; see e.g. Dupačová and Kopa [**10**]. On the other hand, neither coherence, nor convexity of $F(x, P)$ in x are required.

The choice of a degenerated distribution Q concentrated at ω^* with probability 1 corresponds to an additional scenario ω^* and (6) or (8) provide information about *the influence of including the additional scenario* ω^* on the optimal outcome. This helps e.g. to quantify the stress test designed for the optimal performance $\varphi(P)$ of the problem (1) with the underlying probability distribution P obtained by sampling from a theoretical probability distribution or based on observed realizations of ω.

3. Contamination and stress testing for CVaR-related risk measures

In this section we shall consider first the two most popular risk measures, Value at Risk and Conditional Value at Risk.

3.1. Basic formulas for VaR and CVaR

Value at Risk (VaR) was introduced and recommended as a generally applicable risk measure to quantify, monitor and limit financial risks and to identify losses which occur with an acceptably small probability. Unfortunately, the formal definition is inconsistent in the literature. Denote

- $g(x, \omega)$ the loss if $x \in \mathcal{X}$ is selected and realization ω occurs,
- $G(x, P; k) := P\{\omega : g(x, \omega) \le k\}$ the distribution function of the loss associated with a fixed decision $x \in \mathcal{X}$.

Similarly as in Pflug [**17**] or Rockafellar and Uryasev [**20**], in this paper the *Value at Risk* at the confidence level $\alpha \in (0,1)$, VaR_α, is defined as the α-quantile of the loss distribution

$$\mathrm{VaR}_\alpha(x,P) = \min\{k \in \mathbb{R} : G(x,P;k) \geq \alpha\} \tag{9}$$

or

$$\mathrm{VaR}^+_\alpha(x,P) = \inf\{k \in \mathbb{R} : G(x,P;k) > \alpha\}.$$

Hence, a random loss greater than VaR occurs with probability $1-\alpha$. This interpretation is well understood in the financial practice. VaR is not subadditive in general but it can be obtained for an arbitrary probability distribution, existence of moments is not needed. Asymptotic properties of empirical VaR follow from the known results for empirical quantiles, see e.g. Serfling [**22**]. $\mathrm{VaR}(x,P)$ is not concave in P which means that only local contamination bounds can be constructed, cf. Dupačová and Polívka [**9**].

The *Conditional Value at Risk* (CVaR_α) is the mean of the tail distribution G_α of $g(x,\omega)$ defined as

$$G_\alpha(x,P;k) = \begin{cases} 0 & \text{for } k < \mathrm{VaR}_\alpha(x,P) \\ \dfrac{G(x,P;k)-\alpha}{1-\alpha} & \text{for } k \geq \mathrm{VaR}_\alpha(x,P). \end{cases} \tag{10}$$

Assume that $E_P|g(x,\omega)| < \infty$ for all $x \in \mathcal{X}$ and define

$$\Phi_\alpha(x,\psi,P) = \psi + \frac{1}{1-\alpha} E_P(g(x,\omega)-\psi)^+, \tag{11}$$

where $(a)^+ = \max\{a,0\}$. The fundamental minimization formula of Rockafellar and Uryasev [**20**] helps to evaluate CVaR and to analyze its stability including stress testing.

Theorem 1. As a function of ψ, $\Phi_\alpha(x,\psi,P)$ is finite and convex,

$$\min_\psi \Phi_\alpha(x,\psi,P) = \mathrm{CVaR}_\alpha(x,P) \tag{12}$$

and the set of optimal solutions is the interval

$$I(x,P) = \arg\min_\psi \Phi_\alpha(x,\psi,P) = [\mathrm{VaR}_\alpha(x,P), \mathrm{VaR}^+_\alpha(x,P)]. \tag{13}$$

For fixed x, the auxiliary function $\Phi_\alpha(x,\psi,P)$ is linear in P and convex in ψ, hence, $\mathrm{CVaR}_\alpha(x,P)$ is concave in P. Its derivative equals

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{CVaR}_\alpha(x,P_t)|_{t=0^+} = \min_{\psi \in I(x,P)} \Phi_\alpha(x,\psi,Q) - \mathrm{CVaR}_\alpha(x,P), \tag{14}$$

compare with (7). These properties open the possibility of stress testing via contamination for CVaR and for optimization problems with the $\mathrm{CVaR}(x,P)$ objective function as done in Dupačová and Polívka [**9**]. For asymptotic properties of the empirical CVaR see section 6.5 of Shapiro et al. [**24**].

3.2. Example: Stress testing for CVaR

Let P be a discrete probability distribution concentrated on $\omega^1, \dots, \omega^S$ with probabilities $p_s > 0$, $s = 1, \dots, S$, $\sum_s p_s = 1$ and x a *fixed element* of $\mathcal{X}$. Then the program (12) has the form

$$\min_{\psi} \psi + \frac{1}{1-\alpha} \sum_s p_s (g(x, \omega^s) - \psi)^+ \tag{15}$$

and can be further rewritten as

$$\min_{\psi, y_1, \dots, y_S} \{\psi + \frac{1}{1-\alpha} \sum_s p_s y_s : y_s \geq 0,\ y_s + \psi \geq g(x, \omega^s) \text{ for all } s\}.$$

Consider now a stress test of $\mathrm{CVaR}_\alpha(x, P)$, i.e., of the optimal value of (15) whose objective function is linear in P. Let $\psi^* = \psi^*(x, P)$ be an optimal solution of (15), ω^* the stress scenario and Q the corresponding degenerated probability distribution. To apply the contamination technique we proceed as explained in Section 2 for the objective function linear in P. As a result, we obtain contamination bounds

$$\begin{aligned} &(1-t)\,\mathrm{CVaR}_\alpha(x, P) + t\,\mathrm{CVaR}_\alpha(x, Q) \leq \mathrm{CVaR}_\alpha(x, P_t) \\ &\leq (1-t)\,\mathrm{CVaR}_\alpha(x, P) + t\Phi_\alpha(x, \psi^*, Q) \\ &= \Phi_\alpha(x, \psi^*, P_t) \end{aligned} \tag{16}$$

that are valid for all $t \in [0, 1]$; compare with (8).

As the next step, let us discuss briefly *optimization problems with the* $\mathrm{CVaR}_\alpha(x,P)$ *objective function*

$$\min_x \mathrm{CVaR}_\alpha(x, P) \text{ on a fixed compact nonempty set } \mathcal{X} \in \mathbb{R}^n.$$

Substituting (12) for $\mathrm{CVaR}_\alpha(x, P)$, the problem becomes again one with the expectation-type objective function:

$$\min_{x,\psi} \{\Phi_\alpha(x, \psi, P) : x \in \mathcal{X}\}. \tag{17}$$

For convex $\mathcal{X}$ and convex loss functions $g(\bullet, \omega)$ for all ω, $\Phi_\alpha(x, \psi, P)$ is convex in (x, ψ) and standard stability results apply. Moreover, if P is a discrete probability distribution, $g(\bullet, \omega)$ a linear function of x and $\mathcal{X}$ convex polyhedral, we obtain a *linear program*

$$\min_{\psi, y_1, \dots, y_S, x} \Big\{\psi + \frac{1}{1-\alpha} \sum_s p_s y_s : y_s \geq 0, x^\top \omega^s - \psi - y_s \leq 0 \text{ for all } s, x \in \mathcal{X}\Big\}. \tag{18}$$

Let $\psi^*(P)$, $x^*(P)$ be an optimal solution of (17) and denote $\varphi_{C_\alpha}(P)$ the optimal value of (17). The contamination bounds follow the usual pattern, compare with (8), (16)

$$\begin{aligned} &(1-t)\varphi_{C_\alpha}(P) + t\varphi_{C_\alpha}(Q) \\ &\leq \varphi_{C_\alpha}(P_t) \leq (1-t)\varphi_{C_\alpha}(P) + t\Phi_\alpha(x^*(P), \psi^*(P), Q) \end{aligned}$$

for all $t \in [0, 1]$. To apply them one has to evaluate $\Phi_\alpha(x^*(P), \psi^*(P), Q)$ and to solve (17) for the stress distribution Q. We refer to Dupačová and Polívka [**9**] for a numerical example with discrete probability distributions P, Q .

Applicability of contamination bounds depends on tractability of numerical evaluation of the derivative $\varphi'_Q(0^+)$, i.e., of $\Phi_\alpha(x^*(P), \psi^*(P), Q)$, at the already known optimal solution of (17). This is a relatively simple task for static models with a discrete probability distribution P involving CVaR or optimal CVaR, but might become quite hard when constructing contamination bounds for large scale multistage stochastic programs with multiperiod versions of CVaR objective; see Dupačová and Kozmík [**11**].

3.3. Example: Multiperiod CVaR

Consider $z = (z_1, \ldots, z_T)$ stream of random outputs, with z_τ based on a sequence of past observations and decisions up to time τ. Think of a weighted average of risk measures constructed for each time step separately, calculated for each z_τ using the marginal probability distributions P_τ. However, such procedure would not take the information structure of the problem into account. A multiperiod risk measure ρ should consider whole random process z along with its structure and produce a real number based on its distribution.

Using *conditional* distributions P^c_τ given by random information preceding the time step τ, we employ a mean-CVaR risk operator. For time step τ and $0 \le \lambda_\tau \le 1$ we have:

$$\rho_{\tau,\alpha_\tau}(z, P^c_\tau) = (1 - \lambda_\tau) E_{P^c_\tau}[z_\tau] + \lambda_\tau \, \mathrm{CVaR}_{\alpha_\tau}(z_\tau, P^c_\tau). \tag{19}$$

The risk value $\rho_{\tau,\alpha_\tau}(z, P^c_\tau)$ is a random variable, conditional on the state in the step τ, and the final risk measure is obtained by applying expectation to risk values of each time step (cf. Pflug and Römisch [**18**]):

$$\rho(z, P) = \sum\nolimits_{\tau=1}^{T} \mu_\tau E[\rho_{\tau,\alpha_\tau}(z, P^c_\tau)], \tag{20}$$

with $\sum_{\tau=1}^{T} \mu_\tau = 1$, $\mu_\tau \ge 0$ for all τ. This form of $\rho(z, P)$ is particularly friendly as its optimization can be decomposed to dynamic programming equations, which allow to compute the cost function of each stage independently. Moreover, when using this risk measure to form the objective function $F(x, P)$ in (1), the set of feasible first-stage decisions remains independent of P. We refer to Dupačová and Kozmík [**11**] for other CVaR related multiperiod risk measures.

In general, multistage stochastic programs are very hard to solve and a discrete approximation of the true distribution P has to be built. This approximating distribution is represented by a scenario tree and we are able to form a single deterministic optimization problem, which considers all nodes of this tree. However, such problems are usually too large to be solved by the standard software and we need to exploit the special structure to obtain efficient algorithms. In particular, our risk-averse multistage programs have convex cost functions which can be approximated by polyhedral functions from below. We solve our models using the stochastic dual dynamic programming algorithm (SDDP) which originated in the work of Pereira and Pinto [**16**]. SDDP-style algorithms rely on the assumption of stage-wise independence to provide good performance for problems with multiple stages. The algorithm performs series of forward and backward iterations until

a satisfactory solution is found, meaning that some stopping rule, given lower and upper bound, is fulfilled; see Kozmík and Morton [**14**] and references therein. Computation of the upper bound is straightforward in the risk-neutral case, but it has to be enhanced to handle the case of CVaR. We applied the importance sampling procedure derived in Kozmík and Morton [**14**].

During a typical iteration of the SDDP algorithm, cuts have been accumulated at each stage. These cuts represent a piece-wise linear approximation of the future cost functions. On a forward pass we sample a number of linear paths through the tree. As we solve a sequence of problems along these forward paths, the cuts that have been accumulated so far are used to form decisions at each stage. The costs incurred along all the sampled forward paths through the tree can be used to estimate the expected cost of the current policy, thus providing the upper bound.

In the backward pass of the algorithm, we add cuts to the collection defining the current approximation of the future cost functions. We do this by solving programs corresponding to the descendant nodes of each node in the linear paths from the forward pass, except in the final stage T. To form a cut, we use the objective values and subgradients of the descendant nodes to calculate the subgradient of the future cost function. The cuts collected at any node in stage τ apply to all the nodes in that stage, therefore only one set of cuts is maintained for each stage. This complexity reduction is possible because of the stage-wise independence assumption. The optimal value of the first-stage problem provides the lower bound. We refer to articles by Philpott and Matos [**19**] and Shapiro [**23**] for a thorough description and details about the SDDP algorithm.

3.3.1. Numerical illustration. Our procedures will be demonstrated on a simple asset allocation model. At stage τ, the decisions x_τ denote the allocations (in units of a multiple of a base currency, say CZK), and p_τ denotes gross return per stage; i.e., the ratio of the price at stage τ to that in stage $\tau - 1$. These represent the only random parameters in the model. Our model gives user the possibility to choose risk aversion coefficient $\lambda_\tau \in [0, 1]$ and confidence level $\alpha_\tau \in (0, 1)$ separately for each stage of the model. The assets can be rebalanced at every time step of the model, which invests a total portfolio value of $p_\tau^\top x_{\tau-1}$. We consider the case in which transaction costs are proportional to the value of the assets sold or bought; the fee is f_τ. Our problem at stage τ is of the form familiar from dynamic programming. Linearizing the CVaR term in (19) we obtain its final formulation:

$$
\begin{aligned}
&\mathsf{Q}_\tau(x_{\tau-1}, \psi_{\tau-1}, p_\tau) \\
&= \min_{x_\tau, \psi_\tau, q_\tau, c_\tau} -(1-\lambda_\tau)\mathbf{1}^\top x_\tau + \lambda_{\tau+1}\psi_\tau + \frac{\lambda_\tau}{1-\alpha_\tau} q_\tau + E\left[\mathsf{Q}_{\tau+1}(x_\tau, \psi_\tau, p_{\tau+1})\right] \\
\text{s.t. } &\mathbf{1}^\top x_\tau + f_\tau \mathbf{1}^\top c_\tau = p_\tau^\top x_{\tau-1}, \qquad c_\tau - x_\tau \geq -x_{\tau-1}, \qquad c_\tau + x_\tau \geq x_{\tau-1}, \\
&q_\tau \geq -\mathbf{1}^\top x_\tau - \psi_{\tau-1}, \qquad q_\tau \geq 0, \qquad x_\tau \geq 0,
\end{aligned}
$$

with $\mathsf{Q}_{T+1}(\cdot) \equiv 0$ and $\lambda_{T+1} \equiv 0$.

In the first stage the initial capital is assumed to be equal to 1 and because $-\mathbf{1}^\top x_1$ is then identically -1, we drop the constant from the objective function. The first stage optimal solution is then given by the following program:

$$\begin{aligned} &\min_{x_1,\psi_1} \ \lambda_2\psi_1 + E\left[\mathsf{Q}_2(x_1,\psi_1,p_2)\right] \\ &\text{s.t. } \mathbf{1}^\top x_1 = 1, \qquad x_1 \geq 0. \end{aligned} \tag{21}$$

We have used monthly price data of the 12 most important assets traded on Prague Stock Exchange, January 2009 to February 2012. We have fitted a multidimensional correlated log-normal distribution to the price ratios to obtain the initial distribution P. The contaminating distribution Q was then constructed from P by increasing the variance by 20% to provide a stress test with respect to perturbed volatilities. Balanced scenario trees were constructed by sampling from these distributions, using the polar method for normal distribution sampling. The CVaR levels α_τ were set to 95%. The model was evaluated with risk coefficients $\lambda_\tau = 0.1$ in the case with transaction costs of 0.3%. We have computed the contamination bounds for problems with 3 and 5 stages. In Table 1 we show the setup for the scenario trees used in our algorithm.

stages	**descendants per node**	**total scenarios**
3	1 000	1 000 000
5	1 000	10^{12}

Table 1. Testing problems setup.

The three-stage problems can be solved to optimality using our SDDP algorithm, meaning that there is no gap between the lower bound and the upper bound. For these optimal solutions, standard contamination bounds presented in Section 3.2 can be applied; see Figure 1. Due to the complexity of five-stage problems, we are unable to compute exact solutions and we can only provide contamination bounds based on the lower and upper bounds from the SDDP algorithm. With growing number of stages, the number of nodes to be visited to calculate a precise upper bound quickly exceeds a manageable threshold. The upper bounds and derivatives are estimated by simulation. We have used approximately 10 000 nodes for every estimator of the upper bound and contamination derivative. The bounds are presented in Figure 2.

The results show that we are able to obtain tight contamination bounds in all of our testing setups. With 3 stages we have the spread of 0.06%, considering $t = 50\%$ contamination. For large-scale problems, we obtained the spread of 1.06% for statistically valid bound and 0.41% for the mean of the upper bounds, respectively. The straightforward interpretation of our results would state that the model can be considered stable with respect to growing variance of the underlying random distribution which drives the asset price evolution.

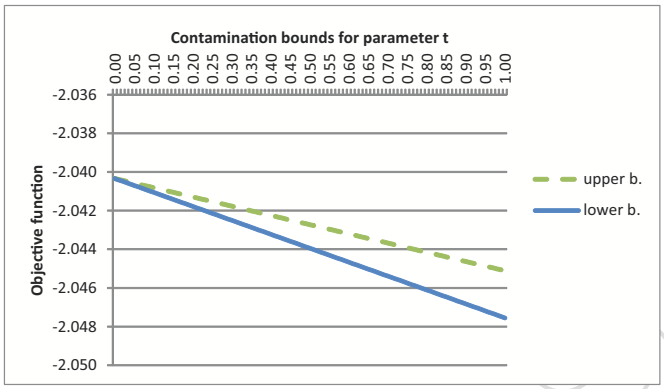

Figure 1. Contamination bounds for a three-stage problem.

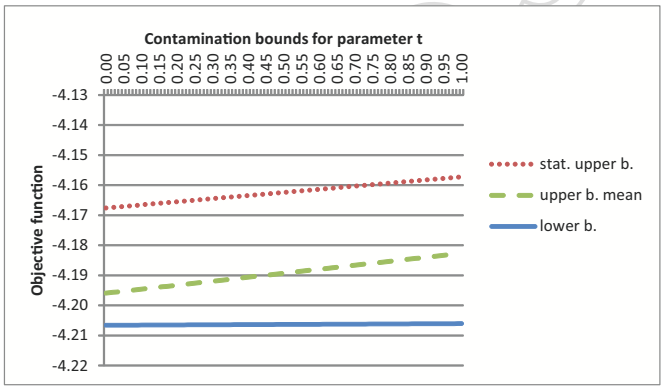

Figure 2. Contamination bounds for a five-stage problem.

4. Spectral risk measures and other extensions

It turns out that CVaR serves as an important building block for other law invariant coherent risk measures: $\mathrm{CVaR}_\alpha(x, P)$ is concave in P and continuous in α on $(0, 1)$, cf. Rockafellar and Uryasev [**20**], and $\sum_i p_i \,\mathrm{CVaR}_{\alpha_i}(x, P)$ with $p_i \geq 0\,\forall i$, $\sum_i p_i = 1$ is also a law invariant coherent risk measure. Its empirical estimate $\sum_i p_i \,\mathrm{CVaR}_{\alpha_i}(x, P_\nu)$ evidently inherits the asymptotic properties valid for the empirical CVaR's with *fixed* individual levels α_i. This type of risk measures was introduced in papers of Acerbi, e.g. [**1**], under the name *spectral risk measures* and was generalized as

$$R_m(x, P) := \int_0^1 \mathrm{CVaR}_\alpha(x, P)\mathrm{d}m(\alpha) \tag{22}$$

with general probability distributions $m(\alpha)$ of α on interval $[0,1]$. Spectral risk measures are concave in P and probability distribution $m(\alpha)$ may reflect the subjective risk aversion of the investor, who is interested in results for alternative confidence levels α.

Consider now a *fixed discrete probability distribution* $m(\alpha)$ and assume for simplicity that $\mathrm{VaR}_{\alpha_i}(x,P) = \mathrm{VaR}^+_{\alpha_i}(x,P)$ for all its atoms α_i, $i = 1,\dots,I$. Denote $\psi^*_{\alpha_i}(x,P) := \mathrm{VaR}_{\alpha_i}(x,P)$. By averaging contamination bounds (16) for $\mathrm{CVaR}_{\alpha_i}(x,P)$ with respect to distribution $m(\alpha)$, bounds for contaminated spectral measures $R_m(x,P_t) = \sum_i p_i \,\mathrm{CVaR}_{\alpha_i}(x,P_t)$ follow:

$$\begin{aligned}(23)\qquad &(1-t)R_m(x,P) + tR_m(x,Q)\\ &\le R_m(x,P_t) \le (1-t)R_m(x,P) + t\sum\nolimits_i p_i \Phi_{\alpha_i}(x,\psi^*_{\alpha_i}(x,P),Q).\end{aligned}$$

For a discrete probability distribution $m(\alpha)$, minimization of $R_m(x,P)$ on the set $\mathcal{X}$ is evidently a problem of multiobjective optimization which provides a solution $x^*_m(P)$ efficient with respect to all involved $\mathrm{CVaR}_{\alpha_i}(x,P)$, $i = 1,\dots,I$, objectives. To get it one may use an optimization shortcut similar to (17), i.e. to solve

$$(24)\qquad \min_{x,\psi_1,\dots,\psi_I} \Big\{ \sum_i p_i \Phi_{\alpha_i}(x,\psi_i,P) : x \in \mathcal{X} \Big\}.$$

This is again a problem with an expectation-type of objective. Hence, contamination bounds for $\min_x R_m(x,P_t)$ can be constructed similarly as for optimization problems with the $\mathrm{CVaR}_\alpha(x,P_t)$ objective function using the general pattern (8). The upper contamination bound exploits optimal solutions $x^*_m(P)$, $\psi^*_i(P)$, $i = 1,\dots,I$ of (24).

Example. In our numerical illustrations presented in Section 3.3.1, replace the risk measure (19) with a risk measure that is based on two CVaR values. For the time step τ and $0 \le \lambda_\tau \le 1$ we have:

$$\begin{aligned}(25)\qquad &\rho_{\tau,\alpha_\tau}(z,P^c_\tau)\\ &= (1-\lambda_\tau-\theta_\tau)E_{P^c_\tau}[z_\tau] + \lambda_\tau \,\mathrm{CVaR}_{\alpha_\tau}(z_\tau,P^c_\tau) + \theta_\tau \,\mathrm{CVaR}_{\beta_\tau}(z_\tau,P^c_\tau).\end{aligned}$$

We have applied the SDDP algorithm with a slight modification to cover the second CVaR term in the objective function. The results for the three-stage problem with CVaR levels $\alpha_\tau = 95\%$ and $\beta_\tau = 99\%$ and risk constants of $\lambda_\tau = 0.05$ and $\theta_\tau = 0.05$ show similar shape of the contamination bounds as in the case with a single CVaR level. The spread is again very small, 0.05% for the $t = 50\%$ contamination.

Spectral risk measures with a continuous probability distribution $m(\alpha)$ offer various interesting theoretical problems. A proof of consistency of empirical spectral risk measures for a general probability distribution $m(\alpha)$ can be found in Acerbi [**1**]. With a fixed probability distribution P one can study the influence of investor's choice of m in (22) via contamination of m by another fixed probability distribution $\hat{m}$ on $[0,1]$.

A further generalization concerns the law invariant coherent risk measures $R(x, P)$ written in the form of the *Kusuoka representation*: Using dual representation of coherent risk measures it is possible to write these risk measures as

$$R(x,P) = \sup\Big\{ \int_0^1 \mathrm{CVaR}_\alpha(x,P)\mathrm{d}m(\alpha) : m \in \mathcal{M}\Big\} \tag{26}$$

where $\mathcal{M}$ is a family of probability distributions on $[0, 1)$. See Kusuoka [**15**] or Shapiro et al. [**24**, Theorems 6.24] and Pflug and Römisch [**18**, 2.45] for details and proof.

Let us discuss contamination bounds for the special case of $\mathcal{M}$ consisting of a finite number of probability measures $m_j(\alpha), j = 1, \dots, J$. In general the contaminated risk measure

$$R(x,t) := R(x,P_t) = \sup\{R_m(x,P_t) : m \in \mathcal{M}\}$$

is not concave in t. However there is $m^* \in \mathcal{M}$ and $t_0 > 0$ such that $R(x,t) = R_{m^*}(x,P)$ for $0 \le t \le t_0$, hence $R(x,t)$ is concave for $0 \le t \le t_0$. The derivative $\frac{\mathrm{d}}{\mathrm{d}t}R_{m^*}(x,t)|_{t=0^+}$ determines the slope of the local upper contamination bound for $R(x,t)$, $0 \le t \le t_0$. The lower bound can be based on an arbitrary spectral risk measure $R_{m_j}(x,P)$, $m_j \in \mathcal{M}$

$$(1-t)R_{m_j}(x,P) + tR_{m_j}(x,Q) \le R_{m_j}(x,P_t) \le R(x,P_t) \qquad \text{for all } t \in [0,1].$$

Evidently, optimization of $R(x,t)$ with respect to $x \in \mathcal{X}$ brings additional problems. For a finite set $\mathcal{M}$, it is possible to evaluate the lower bound $L(P) := \max_j \min_x R_{m_j}(x,P) \le \varphi(P) = \min_x \max_j R_{m_j}(x,P)$ and for an arbitrary $\hat{x} \in \mathcal{X}$, the upper bound $\varphi(P) \le U(P) := \max_j R_{m_j}(\hat{x},P) = R_{\hat{m}}(\hat{x},P)$. The local lower contamination bound for $L(P_t) = \min_x R_{m^*}(x,P_t)$ for $0 \le t \le t_0$ and an upper contamination bound for $U(P_t) = R_{\hat{m}}(\hat{x},P_t)$ can be constructed similarly as for the spectral risk measures. However, the final bounds will be hardly tight enough to provide a useful information about the magnitude of $\varphi(P_t)$.

At this place advantages of *polyhedral risk measures* (cf. Eichhorn and Römisch [**12**]) become apparent. Briefly, the polyhedral risk measure is defined as the optimal value of a certain stochastic linear program with fixed complete recourse for which dual feasibility condition is fulfilled. The only random coefficient $z(\omega)$ appears on the right-hand side. Polyhedral risk measures are concave in P. To obtain a *coherent* polyhedral risk measure, parameters must be appropriately chosen. General criteria on their choice are based on conjugate duality; cf. Eichhorn and Römisch [**12**, Corollary 2.5], and Pflug and Römisch [**18**, Proposition 2.6, or Proposition 2.68]. CVaR is a special example of coherent polyhedral risk measure.

The important property is that minimization of a stochastic program with a polyhedral risk measure in the objective can be transformed to minimization of a common stochastic linear program with a linear expectation-type objective extended for additional variables and constraints, see Proposition 4.1 of Eichhorn and Römisch [**12**]. The contamination technique both for polyhedral risk measures and for two-stage stochastic optimization problems based on them is detailed in Dupačová [**6**] and applied to a scenario-based stochastic program. The definition

and results concerning multiperiod polyhedral risk measures can be found e.g. in Eichhorn and Römisch [**12**] and in Pflug and Römisch [**18**]; for contamination bounds see Dupačová et al. [**8**].

5. Conclusions

Contamination technique provides numerically tractable bounds for static stochastic programs with risk-averse objectives which aim at quantification of changes in the optimal value due to perturbations of the assumed probability distribution. It is convenient for scenario-based problems where it is applicable in stress testing with respect to additional scenarios. Using additional variables and constraints, the majority of the discussed risk measures can be transformed to an expectation form which simplifies the construction of contamination bounds. Variance is an example of a risk measure concave in the probability distribution P for which such transformation does not hold; for the corresponding contamination bounds see e.g. Dupačová [**7**].

The contamination technique can be extended to multistage problems with multiperiod risk measures. We provide numerical results for a simple asset allocation model with the multiperiod CVaR. The contamination technique exhibits good performance and provides sufficiently tight bounds that can be used in practical applications to test stability. For large-scale problems, contamination bounds have to be approximated, but even in this case our procedures can provide statistically valid bounds, which are reasonably tight.

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References

1. Acerbi, C., *Spectral measures of risk: A coherent representation of subjective risk aversion*, J. Banking & Finance **26** (2002), 1505–1518.
2. Artzner, P., Delbaen, F., Eber, J. M. and Heath, D., *Coherent measures of risk*, Mathematical Finance **9** (1999), 203–228.
3. Bonnans, J. F. and Shapiro, A., *Perturbation Analysis of Optimization Problems*, Springer, New York, 2000.
4. Dupačová, J., *Stability and sensitivity analysis in stochastic programming*, Ann. Oper. Res. **27** (1990), 115–142.
5. ———, *Scenario based stochastic programs: Resistance with respect to sample*, Ann. Oper. Res. **64** (1996), 21–38.
6. ———, *Risk objectives in two-stage stochastic programming models*, Kybernetika **44** (2004), 227–242.
7. ———, *Stress testing via contamination*, in: K. Marti et al. (eds.) *Coping with Uncertainty. Modeling and Policy issues*. LNEMS 581, Springer (2006), pp. 29–46.
8. Dupačová, J., Bertocchi, M. and Moriggia, V., *Testing the structure of multistage stochastic programs*, Comput Manag Sci **6** (2009), 161–185.
9. Dupačová, J. and Polívka, J., *Stress testing for VaR and CVaR*, Quantitative Finance **7** (2007), 411–421.
10. Dupačová, J. and Kopa, M., *Robustness in stochastic programs with risk constraints*, Ann. Oper. Res. **200** (2012), 55–74.

11. Dupačová, J. and Kozmík, V., *Structure of risk-averse multistage stochastic programs*, OR Spectrum, doi: 10.1007/s00291-014-0379-2 (2014).
12. Eichhorn, A. and Römisch, W., *Polyhedral risk measures in stochastic programming*, SIAM J. Optimization **16** (2005), 69–95.
13. Gol'shtein, E. G., *Vypukloje Programmirovanije. Elementy Teoriji*, Nauka, Moscow (1970); [*Theory of Convex Programming*, Translations of Mathematical Monographs, **36**, American Mathematical Society, Providence RI, 1972].
14. Kozmík, V. and Morton, D., *Evaluating policies in risk-averse multi-stage stochastic programs*, Mathematical Programming, doi: 10.1007/s10107-014-0787-8 (2014).
15. Kusuoka, S., *On law invariant risk measures*, Advances in Mathematical Economics **3** (2001), 83–95.
16. Pereira, M. V. F. and Pinto, L. M. V. G.: *Multi-stage stochastic optimization applied to energy planning*, Mathematical Programming **52** (1991), 359–375.
17. Pflug, G. Ch., *Some remarks on the Value-at-Risk and the Conditional Value-at-Risk*, in: Uryasev, S. (ed.) *Probabilistic Constrained Optimization, Methodology and Applications.* Kluwer Acad. Publ., Dordrecht (2001), pp. 272–281.
18. Pflug, G. Ch. and Römisch, W., *Modeling, Measuring and Managing Risk.* World Scientific, New Jersey, 2007.
19. Philpott, A. B. and de Matos, V. L.: *Dynamic sampling algorithms for multi-stage stochastic programs with risk aversion*, European Journal of Operational Research **218** (2012), 470–483.
20. Rockafellar, R. T. and Uryasev, S., *Conditional value-at-risk for general loss distributions*, J. of Banking and Finance **26** (2001), 1443–1471.
21. Römisch, W., *Stability of stochastic programming problems*, in: Ruszczyński, A. and Shapiro, A. (eds.) *Handbook on Stochastic Programming.* Elsevier, Amsterdam (2003), pp. 483–554.
22. Serfling, R. J., *Approximation theorems in mathematical statistics.* Wiley, New York, 1980.
23. Shapiro, A., *Analysis of stochastic dual dynamic programming method*, European Journal of Operational Research **209** (2011), 63–72.
24. Shapiro, A., Dentcheva, D. and Ruszczyński, A., *Lectures on Stochastic Programming: Modeling and Theory*, SIAM-Society for Industrial and Applied Mathematics, Philadelphia, 2009.

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