POSTOPTIMALITY FOR SCENARIO BASED FINANCIAL PLANNING MODELS WITH AN APPLICATION TO BOND PORTFOLIO MANAGEMENT [1]

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Abstract. The contamination technique is presented as a numerically tractable technique for postoptimality analysis and analysis of the robustness of the optimal value of various scenario based stochastic programs with respect to inclusion of additional "out-of-sample" scenarios. Using results based on the initial selection of scenarios and those based on the alternative out-of-sample scenarios it provides bounds for the optimal value based on the pooled sample of scenarios of these groups. The application of the method to models supporting financial decision making is detailed for bond portfolio management and tracking models. Numerical experience is presented for a bond portfolio management model using data from the Italian bond market.

Keywords: Contamination technique, SLP with random recourse, tracking models, postoptimality, bond portfolio management.

1 Scenario Based Models

The outcome of financial decisions depends on realization of numerous input values which are unknown to the decision maker at the time when the decision has to be taken. Examples include future prices or returns, interest rates, exchange rates, external cashflows including liabilities, prepayment rates, lapse behavior and future

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inflation. Given a set of forecasted values of these parameters from a scenario, one accepts a decision which is plausible under the assumed circumstances but which may be unacceptable for a different scenario. Another approach is to interprete the input parameters as random and to base the decisions on a stochastic programming model; we refer to the recent monographs by Kall and Wallace (1994), and Prékopa (1995) for a general information about stochastic programming and to collections Konno et al., eds. (1993), Zenios, ed. (1993), Zenios and Ziemba, eds. (1992), surveys Dupačová (1991), Mulvey (1994) or to numerous papers on applications of stochastic programming in finance, e.g., Bradley and Crane (1972), Cariño et al. (1994), Dembo (1991), Dempster and Ireland (1988), Dert (1995), Dupačová and Bertocchi (1996), Kusy and Ziemba (1986), McKendall et al. (1994), Mulvey and Vladimirou (1992), Shapiro (1988), and Zenios (1991).

The numerical techniques designed for solving stochastic programming problems are mostly based on approximation of the distribution of the random parameters by a discrete scenario, obtained by sampling in the course of numerical solution or given in advance; cf. Ermoliev and Wets, eds. (1988). We shall consider here the latter approach; hence, we assume that there is a given *discrete* distribution P concentrated in a finite number of points, say, $\omega_1, \ldots, \omega_S$ with positive probabilities $p_s > 0 \quad \forall s, \quad \sum_{s=1}^{S} p_s = 1$, that enter the coefficients and the function values in a known way. The atoms $\omega_1, \ldots, \omega_S$ are called *scenarios*.

The origin of scenarios can be very diverse; they may be from a truly discrete known distribution, be obtained in the course of a discretization/approximation scheme or by a limited sample information, or come from attempts to model uncertainty by means of scenarios obtained by a preliminary analysis of the problem and with probabilities of their occurence that may reflect an ad hoc belief or a subjective opinion of an expert.

One is interested in both the robustness of the obtained optimal solution and the optimal value of the objective function. The procedure should be robust in the sense that small perturbances of the input, i.e., of the chosen scenarios and of their probabilities, should alter the outcome only slightly so that the results obtained remain close to the unperturbed ones, and that somewhat larger perturbations do not cause a catastrophe. The importance of robust procedures increases with the complexity of the model and with its dimensionality.

We shall elaborate here the *contamination technique* which is, inter alia, suitable for analysis of influence of additional scenarios and for constructing the corresponding error bounds. We refer to Dupačová (1986, 1990) for the theoretical results, to Dupačová (1995) for an application in the field of scenario based multistage stochastic linear programs with fixed complete recourse and to Dupačová (1996 b, 1998) for an extension to problems in which the objective function is nonlinear in distribution P for to cover, e.g., the case of mean-variance criterion or the robust optimization models by Mulvey et al. (1995).

The models considered in this paper can be put into the form:

$$\text{Minimize} \quad f(\mathbf{x}, P) \quad \text{on the set} \quad \mathcal{X} \subset R^n \tag{1.1}$$

where

f convex in $\mathbf{x}$ and linear in P;

P is the probability distribution of the random parameters $\omega \in \Omega$ that enter the problem; in the case of scenario based stochastic programs that we deal with in our applications, P is a discrete probability distribution and for a given set Ω of possible scenarios, this distribution is *fully determined* by the vector $\mathbf{p}$ of their probabilities. Accordingly, the objective function is linear in $\mathbf{p}$.

$\mathcal{X}$ a closed, nonempty set that *does not depend on* P; and

$\mathbf{x} \in \mathcal{X}$ the main, scenario independent decision variable, typically, the first stage decision.

Problems with $f(\mathbf{x}, \bullet)$ *linear* in P that are considered in this paper correspond to minimization of the expected value of a random outcome of the modeled decision process.

Example 1. Scenario based two-stage stochastic linear programs (SLP) with *random relatively complete recourse* appear in financial models that take into account random prices in connection with portfolio rebalancing or with conservation of cashflows, cf. Golub et al. (1993), McKendall et al. (1994), Zenios (1991).

They can be written as

$$\text{minimize} \quad \mathbf{c}^\top \mathbf{x} + \sum_{s=1}^{S} p_s \mathbf{q}_s^\top \mathbf{y}_s \tag{1.2}$$

subject to

$$\begin{array}{lllllll}
\mathbf{A}\mathbf{x} & & & & & & = \mathbf{b} \\
\mathbf{T}_1\mathbf{x} & + & \mathbf{W}_1\mathbf{y}_1 & & & & = \mathbf{h}_1 \\
\mathbf{T}_2\mathbf{x} & + & & \mathbf{W}_2\mathbf{y}_2 & & & = \mathbf{h}_2 \\
\vdots & & & & \ddots & & \vdots \\
\mathbf{T}_S\mathbf{x} & + & & \dots & & \mathbf{W}_S\mathbf{y}_S & = \mathbf{h}_S
\end{array}$$

$$\mathbf{x} \geq 0, \mathbf{y}_s \geq 0, s = 1, \dots, S \tag{1.3}$$

where $\omega_s = [\mathbf{q}_s, \mathbf{T}_s, \mathbf{W}_s, \mathbf{h}_s], s = 1, \dots, S$ are scenarios or atoms at which the probability distribution P is concentrated and $p_s \geq 0, s = 1, \dots, S$ are their probabilities, $\sum_s p_s = 1$.

Example 2. Scenario based expected utility models use principle of the maximal expected utility, namely

$$\text{maximize} \quad \sum_{s=1}^{S} p_s u(g(\mathbf{x}, \omega_s)) \tag{1.4}$$

subject to $\mathbf{x} \in \mathcal{X}$. The function g is often defined as the optimal value of an auxilliary optimization problem that is related with a given scenario ω_s and a given initial decision $\mathbf{x}$. This optimal value can be the final wealth achieved by optimal management of a bond portfolio at the end of the pay-off period (see Sections 3 and 4) or the difference between the return of the portfolio and the index, see Worzel et al. (1994), etc. The choice of the utility function is restricted to concave nondecreasing functions and there are various types of utility functions which are popular in finance, such as isoelastic utility functions $u(W) = \frac{W^\gamma}{\gamma}$. The book Ziemba and Vickson (1975) discussed the pros and cons of typical utility functions.

Also the two-stage stochastic linear program from Example 1 can be modified to an expected utility model:

$$\text{minimize} \quad -\sum_{s=1}^{S} p_s u(\mathbf{c}^\top \mathbf{x} + q(\mathbf{x}, \omega_s)) \tag{1.5}$$

on the set $\mathcal{X}$ and with

$$q(\mathbf{x}, \omega_s) = \min_{\mathbf{y}_s} \left\{ \mathbf{q}_s^\top \mathbf{y}_s | \mathbf{W}_s \mathbf{y}_s = \mathbf{h}_s - \mathbf{T}_s \mathbf{x}, \quad \mathbf{y}_s \geq 0 \right\} \tag{1.6}$$

Example 3. The tracking model, see Dembo (1991), related to (2) can be formulated as follows: Let $v_s, s = 1, \ldots, S$ be the optimal values of the *individual* scenario problems

$$\text{minimize} \quad \mathbf{c}^\top \mathbf{x} + \mathbf{q}_s^\top \mathbf{y}_s \tag{1.7}$$

subject to

$$\begin{array}{lcll} \mathbf{A}\mathbf{x} & & & = \mathbf{b} \\ \mathbf{T}_s \mathbf{x} & + & \mathbf{W}_s \mathbf{y}_s & = \mathbf{h}_s \end{array}$$

$$\mathbf{x} \geq 0, \mathbf{y}_s \geq 0. \tag{1.8}$$

Then the basic compromising or tracking model is

$$\text{minimize} \quad \sum_{s=1}^{S} p_s \left(\|\mathbf{c}^\top \mathbf{x} + \mathbf{q}_s^\top \mathbf{y}_s - v_s\| + \|\mathbf{T}_s \mathbf{x} + \mathbf{W}_s \mathbf{y}_s - \mathbf{h}_s\| \right) \tag{1.9}$$

subject to $\mathbf{x}$ and $\mathbf{y}_s \forall s$ that fulfil the "hard" constraints

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{1.10}$$

$$\mathbf{x} \geq 0, \mathbf{y}_s \geq 0, s = 1, \ldots, S. \tag{1.11}$$

The first and second stage solutions obtained by solving this problem track the optimal solutions of the individual scenario problems (1.7)-(1.8) as closely as possible. The norm in (1.9) can be in principle chosen in an arbitrary way; its choice influences the solution procedure.

Further examples that can be used to illustrate the general form of the considered problem (1) and to provide a motivation for our studies are scenario based multistage stochastic programs, see Dupačová (1995).

In these examples, we are interested in resistance of the obtained optimal decisions and of the optimal value with respect to the used input: for the given set of scenarios $\Omega = \{\omega_1, \dots, \omega_S\}$ we want to study the influence of this choice of scenarios ω_s and of their probabilities as well as the influence of inclusion of additional scenarios on the optimal value of the objective function (1.2), (1.4) or (1.9). We exploit classical results of parametric linear and nonlinear programming together with the contamination technique of robust statistics. This is a tractable approach in situations when a straightforward application of standard postoptimality methods of linear programming is in general hardly manageable: even for a fixed sample size S, inclusion of an additional scenario means an extension of the system of equations, for instance those in problem (1.2)-(1.3), for a new block of second-stage constraints and for additional second-stage variables, etc.

In the next Section, we shall briefly describe the contamination technique and provide the main result – bounds on the optimal value of the perturbed problem. This approach will be applied to the bond portfolio management problem which is an application of the expected utility model from Example 2 and to the tracking model of Example 3. Numerical results presented in the last Section are based on an application of the bond portfolio management model to the Italian bond market.

2 Contamination Technique - The Basic Ideas

We present a brief summary of the contamination technique (cf. Dupačová (1986, 1991)) for the general form of stochastic programs (1) under assumptions that $\mathcal{X}$ is a given nonempty convex closed set of feasible solutions that does not depend on the probability distribution P and that the objective function f is convex in $\mathbf{x}$ and linear in P. Let $\varphi(P)$ denotes the minimal value of the objective function in (1) and let $\mathcal{X}(P)$ be the set of optimal solutions. We shall embed the problem (1) into a family of optimization problems parametrized by a *scalar* parameter λ. This family comes from contamination of the original probability distribution P by another *fixed* probability distribution Q, i. e., from using distributions P_λ of the form

$$P_\lambda = (1-\lambda)P + \lambda Q \quad \text{with} \quad \lambda \in (0,1) \tag{2.1}$$

in the objective function of (1.1) at the place of P. For fixed distributions P, Q the contaminated distribution P_λ depends only on λ and

$$f(\mathbf{x}, P_\lambda) := f_Q(\mathbf{x}, \lambda) \tag{2.2}$$

is the corresponding objective function which is a convex - concave function on $R^n \times [0,1]$. Let

$$\varphi(P_\lambda) = \varphi_Q(\lambda) = \inf_{\mathbf{x}\in\mathcal{X}} f_Q(\mathbf{x}, \lambda) \quad \text{and} \quad \mathcal{X}(P_\lambda) = \mathcal{X}_Q(\lambda) = \arg\min_{\mathbf{x}\in\mathcal{X}} f_Q(\mathbf{x}, \lambda) \tag{2.3}$$

be the optimal value function and the set of optimal solutions of the perturbed stochastic program

$$\text{minimize} \quad f(\mathbf{x}, P_\lambda) := f_Q(\mathbf{x}, \lambda) \quad \text{on the set} \quad \mathcal{X}. \tag{2.4}$$

There are various statements about persistence, stability and sensitivity for parametric programs of the above type:

• Under the additional assumption that the set $\mathcal{X}(P) := \mathcal{X}_Q(0)$ of optimal solutions of the original problem (1.1) is nonempty and bounded and that $\mathcal{X}(Q) = \mathcal{X}_Q(1) \neq \emptyset$, the function φ_Q is a finite concave function on $[0, 1]$, continuous at $\lambda = 0$ (cf. Gol'shtein (1972), Theorem 15) and its value at $\lambda = 0$ equals the optimal value of (1.1):

$$\varphi_Q(0) = \min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}, P) = \varphi(P) \tag{2.5}$$

• If the objective function f_Q is jointly continuous with respect to $\mathbf{x}$ and λ, its derivative exists with respect to λ at $\lambda = 0^+$ for all $\mathbf{x}$ from a neighborhood, say, $\mathcal{X}^*$ of $\mathcal{X}(P)$ and if the convergence of the difference quotients $\frac{1}{\lambda}[f_Q(\mathbf{x}, \lambda) - f_Q(\mathbf{x}, 0)]$ for $\lambda \to 0^+$ is uniform in $\mathbf{x}$ on $\mathcal{X}^*$, we can use a slight modification of Theorem 17 of Gol'shtein (1972) to get the derivative of the optimal value of the perturbed program (2.4) at $\lambda = 0^+$:

$$\varphi'_Q(0^+) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi_Q(0^+) = \min_{\mathbf{x} \in \mathcal{X}(P)} \frac{\mathrm{d}}{\mathrm{d}\lambda} f_Q(\mathbf{x}, 0^+). \tag{2.6}$$

When $f(\mathbf{x}, P)$ is linear in P,

$$f_Q(\mathbf{x}, \lambda) = (1 - \lambda) f(\mathbf{x}, P) + \lambda f(\mathbf{x}, Q) \tag{2.7}$$

is a linear function in λ and for an arbitrary fixed $\mathbf{x}$, the sequence of difference quotients is a stationary one. Then, (2.6) reduces to

$$\varphi'_Q(0^+) = \min_{\mathbf{x} \in \mathcal{X}(P)} [f(\mathbf{x}, Q) - f(\mathbf{x}, P)] = \min_{\mathbf{x} \in \mathcal{X}(P)} f(\mathbf{x}, Q) - \varphi(P). \tag{2.8}$$

In this special but important case the derivative equals the difference between the minimal expected cost of an optimal decision based on the initial distribution P if $Q \neq P$ applies and the minimal expected costs under P.

Using (2.8) and concavity of φ_Q on $[0, 1]$ we can bound the considered perturbed optimal value function $\varphi_Q(\lambda)$:

$$(1 - \lambda)\varphi_Q(0) + \lambda\varphi_Q(1) \leq \varphi_Q(\lambda) \leq \varphi_Q(0) + \lambda\varphi'_Q(0^+) \quad \forall \lambda \in [0, 1] \tag{2.9}$$

and get bounds on the relative change of the perturbed optimal value due to contamination:

$$\varphi_Q(1) - \varphi_Q(0) \leq \frac{1}{\lambda}\left[\varphi_Q(\lambda) - \varphi_Q(0)\right] \leq \varphi'_Q(0^+) \quad \forall \lambda \in [0, 1]\,. \tag{2.10}$$

These bounds can be rewritten in terms of the two probability distributions P, Q:

$$(1-\lambda)\varphi(P) + \lambda\varphi(Q) \le \varphi(P_\lambda) \le \varphi(P) + \lambda\varphi'_Q(0^+) \quad \forall\lambda \in [0,1]\,. \tag{2.11}$$

$$\varphi(Q) - \varphi(P) \le \frac{1}{\lambda}\left[\varphi(P_\lambda) - \varphi(P)\right] \le \varphi'_Q(0^+) \quad \forall\lambda \in [0,1]\,. \tag{2.12}$$

The bounds (2.11) and (2.12) are based on the assumed properties of the objective function $f(\mathbf{x}, P)$ as a function of the probability distribution P *without any convexity assumptions concerning random coefficients* that enter the initial formulation of the analyzed stochastic program, such as (1.2)-(1.3), (1.4), (1.5)-(1.6) or (1.7)-(1.11).

When (1.1) has a unique optimal solution, say $\mathbf{x}(P)$ for the initial distribution P, the derivative (2.8) and the bounds (2.11) have the form

$$\varphi'_Q(0^+) = f(\mathbf{x}(P), Q) - \varphi(P) \tag{2.13}$$

$$(1-\lambda)\varphi(P) + \lambda\varphi(Q) \le \varphi(P_\lambda) \le (1-\lambda)\varphi(P) + \lambda f(\mathbf{x}(P), Q) \quad \forall\lambda \in [0,1] \tag{2.14}$$

so the additional numerical effort consists in solution of the stochastic program based on the alternative distribution Q and in evaluation of the function value of this program at the already known point $\mathbf{x}(P)$. If there are multiple optimal solutions the bounds (2.14) computed at an *arbitrary optimal solution* of the initial problem are valid bounds, but not necessarily the most tight ones.

Similarly, one can approximate the optimal value $\varphi(P_\lambda)$ using the solution $\mathbf{x}(Q)$ and the optimal value $\varphi(Q)$ of $\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x}, Q)$ (provided that the set of optimal solutions $\mathcal{X}(Q) = \mathcal{X}_Q(1)$ is nonempty and bounded):

$$(1-\lambda)\varphi(P) + \lambda\varphi(Q) \le \varphi(P_\lambda) \le \lambda\varphi(Q) + (1-\lambda) f(\mathbf{x}(Q), P) \quad \forall\lambda \in [0,1] \tag{2.15}$$

so that

$$\begin{gathered}(1-\lambda)\varphi(P) + \lambda\varphi(Q) \le \varphi(P_\lambda) \le \\ \min\left\{(1-\lambda)\varphi(P) + \lambda f(\mathbf{x}(P), Q), \lambda\varphi(Q) + (1-\lambda) f(\mathbf{x}(Q), P)\right\}, \\ \forall\lambda \in [0,1]\,.\end{gathered} \tag{2.16}$$

The contamination technique is very flexible and it is a suitable tool for postoptimality analysis in various disparate situations. The choice of a degenerated distribution $Q = \delta(\omega_*) := Q_*$ concentrated at $\omega_* \notin \Omega$ corresponds to an additional scenario and (2.11), (2.12) or (2.14) provide an information about *the influence of including the additional scenario* ω_* on the optimal outcome. Similarly, a degenerated distribution $Q_* = \delta(\omega_*)$ with $\omega_* \in \Omega$ models the case of *increasing probability of scenario* ω_* and so on. The derivatives of the optimal value of the program perturbed by a degenerated contaminating distributions are related to the *influence curve* and they can be used to construct further characteristics of robustness acknowledged in robust statistics, cf. Hampel (1974).

Contamination by a distribution Q on Ω that gives the same expectation $E_Q\omega = E_P\omega$ is helpful in studying resistance with respect to changes of the sample in situations where the corresponding input information - the known fixed expectation of the random parameters ω - is to be preserved; see Dupačová (1996 b).

3 Application To Financial Decision Models

3.1 The bond portfolio management problem.

The objective of the portfolio management model is to maximize the expected utility of the wealth at the end of a given planning period subject to securing the prescribed or uncertain future payments. An *active trading strategy,* which allows for rebalancing the portfolio, is permitted under constraints on conservation of holdings for each asset at each time period and on conservation of cashflows. The main factor which determines the prices, cashflows and other coefficients of the model is the evolution of the short term future interest rates. The possible sequences of interest rates can be determined ad hoc or using a probabilistic model of the term structure, e.g., Black et al. (1990). We assume that these scenarios of interest rates have been already selected, indexed by superscripts $s, s = 1, \ldots, S$, and their probabilities fixed as $p_s > 0, s = 1, \ldots, S, \quad \sum_s p_s = 1$.

We follow the notation introduced in Golub et al. (1993), see also Dupačová and Bertocchi (1996):

$j = 1, \ldots, J$ are indices of the considered bonds and T_j the dates of their maturities; the considered horizon for evaluation of prices is $T \geq \max_j T_j$;

$t = 0, \ldots, T_0$ is the considered discretization of the planning horizon;

b_j denote the initial holdings (in face value) of bond j;

b_0 is the initial holding in the riskless asset;

r_t^s is the short term interest rate valid in the time interval $(t, t+1]$ under scenario s;

f_{jt}^s is the cashflow generated from bond j at time t under scenario s expressed as a fraction of the face value;

ξ_{jt}^s and ζ_{jt}^s are the selling and purchasing prices of bond j at time t for scenario s obtained from the corresponding fair prices

$$P_{jt}^s = P_{jt}(\mathbf{r}^s) = \sum_{\tau=t+1}^{T} f_{j\tau}^s \prod_{h=t}^{\tau-1} (1 + r_h^s)^{-1} \tag{3.1}$$

by subtracting or adding fixed transaction costs and spread; the initial prices ξ_{j0} and ζ_{j0} are known, i. e., scenario independent;

L_t is liability due at time t;

x_j/y_j are face values of bond j purchased / sold at the beginning of the planning period, i.e., at $t = 0$, nonnegative *first-stage decision variables*;

z_{j0} is the face value of bond j held in portfolio after the initial decisions x_j, y_j have been made and the auxiliary nonnegative variable y_0^+ denotes the initial surplus.

The second-stage decision variables on rebalancing, borrowing and reinvestment, $x_{jt}^s, y_{jt}^s, z_{jt}^s, y_t^{-s}, y_t^{+s}$ as well as the wealth $W_{T_0}^s$ at the end of the planning horizon depend on scenarios of interest rates.

The model is

$$\text{maximize} \quad \sum_s p_s u(W_{T_0}^s) \tag{3.2}$$

subject to the first-stage constraints on conservation of holdings

$$y_j + z_{j0} = b_j + x_j \quad \forall j \tag{3.3}$$

and on cashflow

$$y_0^+ + \sum_j \zeta_{j0} x_j = b_0 + \sum_j \xi_{j0} y_j \tag{3.4}$$

subject to the second-stage constraints on conservation and holdings for individual interest rate scenarios

$$z_{jt}^s + y_{jt}^s = z_{j,t-1}^s + x_{jt}^s \quad \forall j, s, 1 \le t \le T_0 \tag{3.5}$$

and on cashflow (including rebalancing the portfolio) at each time period $1 \le t \le T_0$

$$\sum_j \xi_{jt}^s y_{jt}^s + \sum_j f_{jt}^s z_{j,t-1}^s + (1 - \delta_1 + r_{t-1}^s) y_{t-1}^{+s} + y_t^{-s} =$$
$$L_t + \sum_j \zeta_{jt}^s x_{jt}^s + (1 + \delta_2 + r_{t-1}^s) y_{t-1}^{-s} + y_t^{+s} \quad \forall s, t \tag{3.6}$$

under nonnegativity of all variables, with $y_0^{-s} = 0 \, \forall s, y_0^{+s} = y_0^+ \, \forall s$ and with

$$W_{T_0}^s = \sum_j \xi_{jT_0}^s z_{jT_0}^s + y_{T_0}^{+s} - \alpha y_{T_0}^{-s} \quad \forall s. \tag{3.7}$$

The multiplier α in (3.7) should be fixed. For instance, a pension plan assumes repeated application of the model with rolling horizon and values $\alpha > 1$ take into account the debt service in the future.

Thanks to the assumed possibility of reinvestments and of unlimited borrowing, the problem has always a feasible solution. The existence of optimal solutions is guaranteed for a large class of utility functions that are *increasing and concave* which will be assumed henceforth. From the point of view of stochastic programming, it is a *scenario based multiperiod two-stage model with random relatively* *complete recourse* and with additional nonlinearities due to the choice of the utility function.

The main output of the model is the optimal value of the objective function (the maximal expected utility of the final wealth) and the optimal values of the first-stage variables x_j, y_j, y_0^+ (and z_{j0}) for all j. They depend on the initial portfolio of bonds, on the model parameters ($\alpha, \delta_1, \delta_2$, transaction costs), on the chosen utility function, on the scheduled stream of liabilities, on the applied model of interest rates and the market data used to fit the model, and on the way how a modest number of scenarios has been selected out of a whole population. If this input is known and an initial trading strategy determined by scenario independent first-stage decision variables x_j, y_j, y_0^+ (and z_{j0}) for all j has been accepted, then the subsequent scenario dependent decisions have to be made in an optimal way regarding the goal of the model. It means that given the values of the first-stage variables y_0^+ and $\mathbf{x}, \mathbf{y}, \mathbf{z}$ with components $x_j, y_j, z_{j0} \forall j$, the maximal contribution of the portfolio management under the sth scenario to the value of the objective function is obtained as the value of the utility function computed for the maximal value of the wealth $W_{T_0}^s$ attainable for the sth scenario under the constraints of the model, i.e., the utility of the optimal value $W_{T_0}^{s*}$ of the linear program

$$\text{maximize } W_{T_0}^s \text{ subject to}$$

$$z_{jt}^s + y_{jt}^s = z_{j,t-1}^s + x_{jt}^s \quad \forall j, 1 \le t \le T_0$$

$$\sum_j \xi_{jt}^s y_{jt}^s + \sum_j f_{jt}^s z_{j,t-1}^s + (1 - \delta_1 + r_{t-1}^s) y_{t-1}^{+s} + y_t^{-s} =$$
$$L_t + \sum_j \zeta_{jt}^s x_{jt}^s + (1 + \delta_2 + r_{t-1}^s) y_{t-1}^{-s} + y_t^{+s}, \quad t = 1, \dots, T_0, \tag{3.8}$$

under nonnegativity of all variables, with $y_0^{-s} = 0, y_0^{+s} = y_0^+, z_{j0}^s = z_{j0} \forall j$ and with

$$W_{T_0}^s = \sum_j \xi_{jT_0}^s z_{jT_0}^s + y_{T_0}^{+s} - \alpha y_{T_0}^{-s}. \tag{3.9}$$

Denote the corresponding maximal value $u(W_{T_0}^{s*})$ of the utility function by $U^s(\mathbf{x}, \mathbf{y}, \mathbf{z}, y_0^+)$. Using this notation we can rewrite the program (3.2)-(3.7) as

$$\text{maximize} \quad \sum_{s=1}^S p_s U^s(\mathbf{x}, \mathbf{y}, \mathbf{z}, y_0^+) \tag{3.10}$$

subject to nonnegativity constraints and subject to (3.3)- (3.4). Except for maximization at the place of minimization, this is already the form which fits the general framework (1.1). The objective function (3.10) is concave in the first-stage decision variables and linear in the initial probability measure P carried by S fixed scenarios indexed as $s = 1, \dots, S$. Denote by $\varphi(P)$ the optimal value of (3.10) and by $\mathbf{x}(P), \mathbf{y}(P), \mathbf{z}(P), y_0^+(P)$ the optimal first-stage decision. For simplicity, assume that this optimal first-stage solution is unique.

Inclusion of other out-of-sample scenarios means to consider a convex mixture of two probability distributions: P that is carried by the initial scenarios indexed by $s = 1, \dots, S$ with probabilities $p_s > 0, \sum_s p_s = 1$ and Q carried by the out-of-sample scenarios indexed by $\sigma = 1, \dots, S'$ with probabilities $\pi_\sigma > 0, \sum_\sigma \pi_\sigma = 1$. Let λ denote the parameter that gives the contaminated distribution

$$P_\lambda = (1-\lambda)P + \lambda Q, \quad 0 \le \lambda \le 1 \tag{3.11}$$

carried by the pooled sample of $S + S'$ scenarios that occur with probabilities $(1-\lambda)p_1, \dots, (1-\lambda)p_S, \lambda\pi_1, \dots, \lambda\pi_{S'}$. For the fixed initial distribution P and a fixed contaminating distribution Q for which the maximal value $\varphi(Q)$ of the objective function $\sum_\sigma \pi_\sigma U^\sigma(\mathbf{x}, \mathbf{y}, \mathbf{z}, y_0^+)$ is finite, the optimal value $\varphi(P_\lambda) := \varphi(\lambda)$ is a finite convex function on [0,1] and its derivative at $\lambda = 0^+$ equals

$$\varphi'(0^+) = \sum_\sigma \pi_\sigma U^\sigma(\mathbf{x}(P), \mathbf{y}(P), \mathbf{z}(P), y_0^+(P)) - \varphi(P) \tag{3.12}$$

cf. (2.13); this should be substituted into the formula (2.11) multiplied by -1 to obtain the bounds for the optimal value $\varphi(P_\lambda)$ for the problem based on the pooled set of $S + S'$ scenarios:

$$(1-\lambda)\varphi(P) + \lambda \sum_\sigma \pi_\sigma U^\sigma(\mathbf{x}(P), \mathbf{y}(P), \mathbf{z}(P), y_0^+(P)) \le \varphi(P_\lambda) \le (1-\lambda)\varphi(P) + \lambda\varphi(Q) \tag{3.13}$$

for all $0 \le \lambda \le 1$. The lower and upper bound coincide if the optimal first-stage solution $\mathbf{x}(P), \mathbf{y}(P), \mathbf{z}(P), y_0^+(P)$ of the initial program (3.2)-(3.7) is optimal also for the corresponding program based on distribution Q carried by the additional S' scenarios indexed by $\sigma = 1, \dots, S'$.

If, for instance, P is carried by S equally probable scenarios (sampled form a given population) and Q is carried by other S' equally probable scenarios sampled from the same population, it is natural to fix λ so that the pooled sample consists of $S + S'$ equally probable scenarios, again

$$\lambda = S'(S + S')^{-1}. \tag{3.14}$$

Hence, the bounds for the optimal value based on the pooled sample of size $S + S'$:

$$S(S+S')^{-1}\varphi(P) + S'(S+S')^{-1} \sum_\sigma \pi_\sigma U^\sigma(\mathbf{x}(P), \mathbf{y}(P), \mathbf{z}(P), y_0^+(P)) \le$$
$$\varphi(P_\lambda) \le S(S+S')^{-1}\varphi(P) + S'(S+S')^{-1}\varphi(Q). \tag{3.15}$$

The additional numerical effort consists in solving the stochastic program

$$\text{maximize} \quad \sum_\sigma \pi_\sigma U^\sigma(\mathbf{x}, \mathbf{y}, \mathbf{z}, y_0^+) \tag{3.16}$$

subject to (3.3)-(3.4) and to nonnegativity constraints for the distribution Q carried by S' out-of-sample scenarios to obtain $\varphi(Q)$ and in evaluation and averaging the S' function values $U^\sigma(\mathbf{x}(P), \mathbf{y}(P), \mathbf{z}(P), y_0^+(P))$ for the new scenarios at the already obtained optimal first-stage solution; these are in fact the main numerical indicators which appear in various simulation studies of the portfolio performance under out-of-sample scenarios, cf. McKendall et al. (1994). For relatively large values of λ (or S'), it pays to use the more complicated lower bound according to (2.16); see Figure 1. The important special case of small λ is $S' = 1$, i. e., the inclusion of one additional scenario.

The bounds can be also used to derive simple rules on the influence of additional scenarios on the optimal value. For instance:

- If the derivative

$$\varphi'(0^+) = \sum_\sigma \pi_\sigma U^\sigma(\mathbf{x}(P), \mathbf{y}(P), \mathbf{z}(P), y_0^+(P)) - \varphi(P) > 0 \tag{3.17}$$

then the optimal value $\varphi(P_\lambda)$ increases for all $0 < \lambda < 1$.

- If $\varphi(Q) < \varphi(P)$, the optimal value $\varphi(P_\lambda)$ decreases at $\lambda = 0^+$.

The postoptimality technique described here is independent of the method which was used to generate or to select the scenarios – the atoms of the distributions P and Q. It can be used without any problems for scenario dependent liabilities and cashflows what allows, for instance, to include the case of callable and puttable bonds and mortgage backed securities (cf. Golub et al. (1993) for the corresponding model formulation).

3.2 The tracking model

Assume that the individual scenario problems (1.7) have been solved for all considered scenarios, i. e., for all sets of coefficients values $\omega_s = [\mathbf{q}_s, \mathbf{W}_s, \mathbf{T}_s, \mathbf{h}_s], s = 1, \ldots, S$ so that the values $v_s \forall s$ needed for the tracking objective function (1.9) are known. The tracking model (1.9) is

$$\text{minimize} \quad \sum_{s=1}^{S} p_s g(\mathbf{x}, \mathbf{y}_s; \omega_s) \tag{3.18}$$

subject to

$$\mathbf{x} \in \mathcal{X} := \{\mathbf{x} | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \tag{3.19}$$

and $\mathbf{y}_s \geq 0 \forall s$, with

$$g(\mathbf{x}, \mathbf{y}_s; \omega_s) = \|\mathbf{c}^\top \mathbf{x} + \mathbf{q}_s^\top \mathbf{y}_s - v_s\| + \|\mathbf{T}_s \mathbf{x} + \mathbf{W}_s \mathbf{y}_s - \mathbf{h}_s\|. \tag{3.20}$$

Let the alternative distribution Q be carried by scenarios $\omega_{S+1}, \ldots, \omega_{S+S'}$ with probabilities $\pi_{S+s} > 0, s = 1, \ldots, S', \sum_s \pi_{S+s} = 1$. Inclusion of additional scenarios brings along new variables $\mathbf{y}_{S+s}$. As we need to work with a fixed set of feasible solutions, one more reformulation is needed before application of the contamination technique discussed in Section 2.

Consider the pooled sample of scenarios $\omega_1, \ldots, \omega_S, \omega_{S+1}, \ldots, \omega_{S+S'}$; the initial distribution P assigns them probabilities $p_s, s = 1, \ldots, S$ and 0 for the remaining ones, the distribution Q assigns zero probabilities to the initial scenarios and probabilities $\pi_{S+s}, s = 1, \ldots, S'$, to the new ones. The perturbed problem carried by the pooled sample is

$$\text{minimize} \quad (1-\lambda)\sum_{s=1}^{S} p_s g(\mathbf{x}, \mathbf{y}_s; \omega_s) + \lambda \sum_{s=1}^{S'} \pi_{S+s} g(\mathbf{x}, \mathbf{y}_{S+s}; \omega_{S+s}) \tag{3.21}$$

subject to nonnegativity of all variables $\mathbf{y}_s, s = 1, \ldots, S + S'$ and subject to (3.19). The derivative of its optimal value at $\lambda = 0^+$ can be computed according to (2.8) as

$$\varphi'_Q(0^+) = \min_{\mathbf{x}^*, \mathbf{y}_s^* \forall s} \sum_{s=1}^{S'} \pi_{S+s} g(\mathbf{x}^*, \mathbf{y}^*_{S+s}; \omega_{S+s}) - \varphi(P). \tag{3.22}$$

The minimization in (3.22) is carried over all optimal solutions of the augmented initial program (3.18)

$$\text{minimize} \quad \sum_{s=1}^{S} p_s g(\mathbf{x}, \mathbf{y}_s; \omega_s) + \sum_{s=1}^{S'} 0 * g(\mathbf{x}, \mathbf{y}_{S+s}; \omega_{S+s}) \tag{3.23}$$

subject to nonnegativity constraints on all variables and subject to (3.19). Due to the special form of (3.22), the minimization concerns the optimal $\mathbf{x}$-part of the solution and arbitrary nonnegative variables $\mathbf{y}_{S+s}, s = 1, \ldots, S'$.

Assume that the optimal $\mathbf{x}$-part of the solution of the initial problem (3.18)-(3.19) is unique, say, $\mathbf{x}(P)$. Then the derivative is obtained by solving S' minimization problems with objective functions

$$g(\mathbf{x}, \mathbf{y}_{S+s}; \omega_{S+s}) = \|\mathbf{c}^\top \mathbf{x}(P) + \mathbf{q}_{S+s}^\top \mathbf{y}_{S+s} - v_{S+s}\| + \|\mathbf{T}_{S+s}\mathbf{x}(P) + \mathbf{W}_{S+s}\mathbf{y}_{S+s} - \mathbf{h}_{S+s}\| \tag{3.24}$$

subject to $\mathbf{y}_{S+s} \geq 0$ and by taking an average of the obtained optimal values with weights π_{S+s}.

In the context of the postoptimality analysis for the tracking models it would be important to get some results concerning the optimal first-stage solutions $\mathbf{x}$. Indeed there exist theoretical results connected with contamination technique (see Dupačová (1986)) and also pathfollowing methods which can be implemented for parametric programs depending on a scalar parameter, as is λ in our case. However, up to now we are not ready to report any numerical experience in this direction.

4 Numerical Results

This Section provides numerical results based on the contamination technique described by formulas (3.13) and (3.15) for the bond portfolio management problem. We analyze the change in the optimal final wealth due to the following typical cases:

- parallel shifts in interest rates scenarios;
- doubling number of scenarios in the sampling strategy;
- doubling number of scenarios when scenarios are randomly generated.

To simulate the behavior of an investment portfolio of fixed income securities on the Italian bond market we use the model described by (3.2)-(3.7) with the linear utility function and within the time horizon of one year ($T_0 = 12$).

The initial portfolio and the term structure are related to September 1, 1994. We consider the same portfolio that was used in Bertocchi et al. (1996) for the sensitivity analysis of portfolio with respect to sampling strategies. It includes typical governmental bonds, paying semi-annual coupons and covering two year forward till 29 years maturities (the so called BTPs) as well as puttable bonds (CTOs), paying semi-annual coupons with the maturity of 8 years and a possible exercise of the option in the 4th year or with the maturity of 6 years and an exercise at the 3rd year; see Table 1.

TABLE 1

Bonds	Qt	coupon	redemp.	payment dates		exercise	maturity
BTP36658	10	3.9375	100.1875	01/04	01/10		01/10/96
BTP36631	20	5.03125	99.5313	01/03	01/09		01/03/98
BTP12687	15	5.25	99.2312	01/01	01/07		01/01/2002
BTP36693	10	3.71875	99.3875	01/08	01/02		01/10/2004
BTP36665	5	3.9375	99.2188	01/05	01/11		01/11/2023
CTO13212	20	5.25	100.0000	20/01	20/07	20/01/95	20/01/98
CTO36608	20	5.25	99.9500	19/05	19/11	19/05/95	19/05/98

Note that the coupon yields and the redemption prices are after tax.

To estimate the term structure of interest rates we used the regression model of Bradley and Crane (1972) applied to the yields obtained by the market quotation of the BTPs on the relevant day; see Figure 2 for the term structure of interest rates. We refer to Dupačová et al. (1996) for a detailed discussion.

In this application, liabilities are not considered, liquidity can be obtained from the interbank market at a rate greater than that one at which surplus can be always reinvested; hence in (3.7) $\alpha = 1$. The additive transaction costs are fixed at ± 0.01, $\delta_2 = 0.025$ and δ_1 is 0 or .001.

The scenarios are based on data from Italian bond market generated according to the Black - Derman - Toy model and selected according to the simplified version of the nonrandom sampling strategy by Zenios and Shtilman (1993) as described in Dupačová and Bertocchi (1996) and in Bertocchi et al. (1996).

The binomial model produces scenarios that can be coded by 2^T ($T = 350$ for our data) binary fractions uniformly distributed in $[0, 1]$ and the sampling strategy chooses the number of periods L for which all possibilities (choices of zeros and ones on the first L positions) are fully covered. The remaining digits necessary to complete the full length paths were selected according to Table 2

TABLE 2

case	$l = 1, ..., L$	$L + 1$	$l = L + 2, ..., T_0$	$T_0 + 1$
B1	$s = 1, ..., 2^3$	$\omega^s_{L+1} = 0$	$\omega^s_l = 0$	$\omega^s_{T_0+1} = 1$
B2	$s = 1, ..., 2^3$	$\omega^s_{L+1} = 0$	$\omega^s_l = 0$	$\omega^s_{T_0+1} = 0$
B3	$s = 1, ..., 2^3$	$\omega^s_{L+1} = 0$	$\omega^s_l = 1$	$\omega^s_{T_0+1} = 1$
B4	$s = 1, ..., 2^3$	$\omega^s_{L+1} = 1$	$\omega^s_l = 1$	$\omega^s_{T_0+1} = 1$
C4	$s = 1, ..., 2^4$	$\omega^s_{L+1} = 1$	$\omega^s_l = 1$	$\omega^s_{T_0+1} = 1$

and for $l > T_0 + 1$, the components ω^s_l alternate up and down (1 or 0) starting with the indicated value of $\omega^s_{T_0+1}$.

We consider two new cases D1 and D2, composed of 8 different paths chosen *randomly* from the uniform distribution on $[0, 1]$ and we add case B2st, with the rates based on the case B2 perturbed by the additive shift of -0.000355 (which corresponds to the shift of 5% of the current B2 rates).

The numerical results reported in the Figures are organized according to the scheme in Table 3.

TABLE 3

distribution P	distribution Q	Figure
B2st	B2	3
B2	B2st	3
B3	B4	4
B4	B3	4
D1	D2	5
D2	D1	5

They consist of the optimal values and optimal initial strategies for the two alternative cases based on distributions P and Q, and they contain the average values $\sum_\sigma \pi_\sigma U^\sigma(\mathbf{x}(P), \mathbf{y}(P), \mathbf{z}(P), y_0^+(P))$ or $\sum_s p_s U^s(\mathbf{x}(Q), \mathbf{y}(Q), \mathbf{z}(Q), y_0^+(Q))$ under headings "means of contam. solutions", the values of the directional derivatives and of the lower and upper bounds computed according to (3.13) for distinct values of λ and a graphical representations of these bounds.

The results in Figures 3 show that the choice of $\delta_1 \neq 0$ for which the return on cash investment is less than that in bonds influences the optimal initial strategy for scenario bed B2, making more valuable the investment in CTO36608 than in cash. For scenarios in B2st the optimal initial strategy does not change. The bounds for $\delta_1 = 0$ are similar to those for $\delta_1 \neq 0$.

Choice of the couple of scenario beds B2 and B2st allows comparison of the situations when a certain bed of scenarios, for example B2st, is changed so that all rates are increased of a fixed quantity, like in B2. The left lower bound and the upper bound in Figures 3 for contamination of B2st by B2 are very precise and show that the optimal final wealth is untouched by a small parallel shift in interest rates. Similar tests of resistance of the optimal value with respect to a shift of interest rates or prices have appeared in several applications, see e.g. McKendall et al. (1994), but without giving the global bounds (2.15) or (2.16) for the optimal value of the perturbed problem.

The results on contamination between B3 and B4, see Figure 4, illustrate another possible application of the bounds, namely, for supporting decisions concerning the required number of scenarios (i. e., concerning the value of L in Table 2). For $\lambda = 0.5$, the example in Figure 4 gives the interval [11350.93, 11351.54] for the optimal value of C4 based on 2^4 scenarios – a union of scenarios from the beds B3 and B4. (Indeed, the true optimal value for C4 is 11350.97.) It means that using the double number of scenarios does not increase essentially the precision of the obtained approximate of the optimal value for the full hypothetical problem which would be based on all 2^{350} possible scenarios of interest rates generated according to the Black - Derman - Toy model. However, the optimal initial investment strategies are quite different.

Finally, Figure 5 illustrates an application of the bounds to the case of the pooled sample based on randomly chosen scenarios, such as in experiments D1 and D2. Again, the bounds for the optimal value based on the pooled sample carried by 16 scenarios, as obtained with $\lambda = 0.5$ from the results of the two small problems for beds D1 and D2 of 8 randomly chosen scenarios, are very tight: [11334, 11334.74]. Exploitation of the more complicated bound of the type of (2.16) helped to increase the lower bound 11331.82 obtained according to (3.13) or (3.15); a similar observation holds true also for Figure 4.

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