Jitka Žáčková

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ON MINIMAX SOLUTIONS OF STOCHASTIC LINEAR PROGRAMMING PROBLEMS

JITKA ŽÁČKOVÁ, Praha

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1. Our starting point is the formulation of a stochastic linear program as a strategic game. This formulation differs only slightly from that given by IOSIFESCU and THEODORESCU [3]. Secondly, we state a minimax theorem for that game and study the methods of solution. In some special but important cases it is shown that the minimax solution of a stochastic linear program is equivalent to the solution of an ordinary linear program (of greater dimension, in general). The existence of a finite solution is also discussed.

2. Let E_n^+ denote the non-negative orthant of the n-dimensional Euclidean space.

Let (A, b, c) – where $A = (a_{ij})$, $b = (b_i)$, $c = (c_j)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ – be a random vector; let its distribution $F(A, b, c)$ belong to a set of distributions $\mathscr{F}$.

Let $r_i(y)$, $i = 1, \ldots, m$ be real functions such that $r_i(y) = 0$ for $y \leqq 0$ and $r_i(y) > 0$ for $y > 0$.

For $x \in E_n^+$, $F \in \mathscr{F}$ set

$$H(x, F) = \mathsf{E}_F \{ \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m} r_i(\sum_{j=1}^{n} a_{ij} x_j - b_i) \} .$$

If H is defined and finite for all $x \in E_n^+$, $F \in \mathscr{F}$, define a two-person zero-sum game by its normal form

$$G = (E_n^+, \mathscr{F}, H) .$$

3. The game G corresponds to the situation, where in the linear program:

$$Ax \leqq b , \quad x \geqq 0 , \quad c'x = \text{maximum} ,$$

A, b, c (or some of them, or some of their components) are random vectors, their simultaneous distribution is known to belong to a set $\mathscr{F}$, the vector x is to be chosen independently of the realization of these random vectors, and the violation of the constraint $\sum_{j=1}^{n} a_{ij}x_j \leqq b_i$ is penalized by the amount

$$r_i\Big(\sum_{j=1}^{n} a_{ij}x_j - b_i\Big), \quad i = 1, \ldots, m .$$

As the *solution* of this program, the optimal pure strategy of the player I in the game G will be meant.

4. For $F \in \mathscr{F}$, let F^* be the corresponding marginal distribution of (A, b); let $\mathscr{F}^* = \{F^* : F \in \mathscr{F}\}$.

Theorem 1. *Suppose that one of the following conditions is satisfied:*

(i) *the set $\mathscr{F}$ is convex and compact (in the sense of Lévy's distance) and the c_j's are uniformly integrable with respect to $F \in \mathscr{F}$,*

(ii) *the set $\mathscr{F}^*$ is convex and compact and $\mathsf{E}_F c$ equals a constant vector for all $F \in \mathscr{F}$.*

Let the functions $r_i(y)$, $i = 1, \ldots, m$, be convex and let the functions $r_i(\sum_{j=1}^{n} a_{ij}x_j - b_i)$ be uniformly integrable with respect to $F \in \mathscr{F}$ for every $x \in E_n^+$.[1])

Then we have

$$\sup_{x \in E_n^+} \min_{F \in \mathscr{F}} H(x, F) = \min_{F \in \mathscr{F}} \sup_{x \in E_n^+} H(x, F) \tag{1}$$

where this common value is either $+\infty$ or it is finite and the suprema can be replaced by maxima.

Proof. Let condition (i) be satisfied. According to FAN KY [2, Theor. 2], it is sufficient to show that H is a continuous and convex function on $\mathscr{F}$ for each $x \in E_n^+$ and that for each element of $\mathscr{F}$ it is a concave function on E_n^+. The continuity of H on $\mathscr{F}$ is a consequence of the uniform integrability of the c_j's and r_i's – see LOÉVE [5, Theor. 11.4A] – and of the special form of H; the convexity is trivial, because H, being an integral with respect to F, is additive and homogeneous in F. The concavity of H on E_n^+ follows easily from the convexity of the r_i's.

Now, let condition (ii) be satified. For every $F \in \mathscr{F}$ let F^* be the mentioned marginal distribution; let γ be the constant vector $\mathsf{E}_F c$. Let us define $H^*(x, F^*) =$

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[1]) The uniform integrability of the r_i's means that for every $x \in E_n^+$ and $i = 1, \ldots, m$,

$$\lim_{N \to \infty} \int \cdots \int_{|a_{ij}| \geqq N, j=1,\ldots,n, |b_i| \geqq N} r_i\Big(\sum_{j=1}^{n} a_{ij}x_j - b_i\Big)\, \mathrm{d}F = 0$$

holds uniformly in $F \in \mathscr{F}$; similarly for the c_j's. Cf. Loève [5, p. 182].

$= \sum_{j=1}^{n} \gamma_j x_j - \mathsf{E}_{F^*}\{\sum_{i=1}^{m} r_i(\sum_{j=1}^{n} a_{ij}x_j - b_i)\}$. It is readily seen that the games $G = (E_n^+, \mathscr{F}, H)$ and $G^* = (E_n^+, \mathscr{F}^*, H^*)$ are equivalent (in the sense of BLACKWELL and GIRSHICK [1, Def. 1.4.2]) and the game G^* again satisfies the conditions of the Fan Ky's theorem.

We shall now investigate the game G for special choices of r_i and $\mathscr{F}$. It will be shown that in some cases the method of solution reduces to an ordinary linear program. As a rule, we shall describe the set of distributions $\mathscr{F}$ in terms of random variables and their properties. In the sequel, Greek letters will always denote known constants (not $\pm\infty$).

5. In the game G, let a_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$, be constants, let b_i, $i = 1, \ldots, m$, be independent random variables such that $\mathsf{E}(b_i) = \beta_i$, $\beta_i' \leqq b_i \leqq \beta_i''$ a.s., $\beta_i' < \beta_i''$; let c_j, $j = 1, \ldots, n$, be random variables such that $\mathsf{E}c_j = \gamma_j$. (Herewith the set $\mathscr{F}$ is defined.) Let $r_i(y) = v_i y^+$, $v_i > 0$, $i = 1, \ldots, m$, where $y^+ = \frac{1}{2}(|y| + y)$.

Let $\Delta = \{I, J, K\}$ be an arbitrary decomposition of the set $\{1, 2, \ldots, m\}$ into three disjoint parts, one or two of which may be empty. Denote Δ_ν, $\nu = 1, \ldots, M$, those of such decompositions, for which the sets

$$(2) \qquad \{x \geqq 0 \colon \sum_{j=1}^{n} a_{ij}x_j \geqq \beta_i'',\ i \in I;\ \beta_i' \leqq \sum_{j=1}^{n} a_{ij}x_j \leqq \beta_i'',\ i \in J;\ \sum_{i=1}^{n} a_{ij}x_j \leqq \beta_i',\ i \in K\}$$

are non-empty. ($M \leqq 3^m$ holds.) Further, define

$$q_{\nu j} = -\gamma_j + \sum_{i \in I_\nu} v_i a_{ij} + \sum_{i \in J_\nu} v_i \lambda_i a_{ij}\,,$$

$$k_\nu = \sum_{i \in I_\nu} v_i \beta_i + \sum_{i \in J_\nu} v_i \lambda_i \beta_i' \quad \nu = 1, \ldots, M\,; \quad j = 1, \ldots, n\,,$$

where $\lambda_i = (\beta_i'' - \beta_i)/(\beta_i'' - \beta_i')$. Finally, denote $Q = (q_{\nu j})$, $\nu = 1, \ldots, M, j = 1, \ldots, n$; $k = (k_\nu)$, $\nu = 1, \ldots, M$; $e' = (1, \ldots, 1)$.

Theorem 2. (i) *Relation* (1) *holds.* (ii) *x is a solution of the game G if and only if (x, y_0) is a solution of the linear program*

$$(3) \qquad Qx + ye \leqq k\,, \quad x \geqq 0\,, \quad y = \text{maximum}\,.$$

Proof. (i) As the b_i are independent it suffices to prove the compactness of the sets of one-dimensional distributions. But this is a consequence of the criterion: $\mathscr{F}$ is compact $\Leftrightarrow$ $\mathscr{F}$ is closed and both $\lim_{t \to -\infty} F(t) = 0$, $\lim_{t \to +\infty} F(t) = 1$ are uniform in $\mathscr{F}$. (See Loève [5, p. 215].) All other conditions of Theorem 1 are evidently satisfied.

(ii) Denote $\varphi(x) = \min_{F \in \mathscr{F}} H(x, F)$. From the independence of b_i's it follows

$$\varphi(x) = \sum_{j=1}^{n} \gamma_j x_j - \sum_{j=1}^{m} v_i \max_{F \in \mathscr{F}} \mathsf{E}_F\{(\sum_{j=1}^{n} a_{ij}x_j - b_i)^+\}\,.$$

The terms of the second sum can be evaluated with the help of a result due to JIŘINA and NEDOMA [4, Theor. 4][1]); we get

$$\varphi(x) = \sum_{j=1}^{n} \gamma_j x_j - \sum_{i=1}^{m} v_i \{\lambda_i (\sum_{j=1}^{n} a_{ij} x_j - \beta'_i)^+ + (1 - \lambda_i)(\sum_{j=1}^{n} a_{ij} x_j - \beta''_i)^+\} .$$

The function $\varphi(x)$ is concave, and it is linear ($\varphi(x) = -\sum_{j=1}^{n} q_{\nu j} x_j + k_\nu$) on each of the sets (2), which correspond to the decompositions Δ_ν, $\nu = 1, \ldots, M$; these sets themselves constitute a (non-overlapping) decomposition of E_n^+. This means that $-\sum_{j=1}^{n} q_{\nu j} x_j + k_\nu$, $\nu = 1, \ldots, M$ are the upper supporting hyperplanes to $\varphi(x)$ and that

$$\varphi(x) = \min_{1 \leqq \nu \leqq M} (-\sum_{j=1}^{n} q_{\nu j} x_j + k_\nu) .$$

Now, the maximization of $\varphi(x)$ is equivalent to the maximization of y under the constraint $y \leqq \varphi(x)$, i.e., under the constraints $y \leqq -\sum_{j=1}^{n} q_{\nu j} x_j + k_\nu$, $\nu = 1, \ldots, M$; but this is the assertion of the theorem.

Theorem 3. *The non-existence of a (strictly) negative column in the matrix Q is a necessary condition and the existence of a non-negative row in Q is a sufficient condition for the existence of a (finite) solution of the game G.*

Proof. The solution of the game G exists $\Leftrightarrow$ there exists the maximum of y on the set $\{x \geqq 0 : Qx + ye \leqq k\}$ (which is always non-empty) $\Leftrightarrow$ there is an admissible vector of the dual linear program to (3), i.e., there exists a vector u, for which

$$u \geqq 0 , \quad \sum_{\nu=1}^{M} u_\nu = 1 , \quad \sum_{\nu=1}^{M} u_\nu q_{\nu j} \geqq 0 , \quad j = 1, \ldots, n .$$

Especially, such a vector exists if there is a non-negative row in Q, and it cannot exist if there is a negative column in Q.

Corollary. *If the set $\{x \geqq 0 : \sum_{j=1}^{n} a_{ij} x_j \geqq \beta''_i,\ i = 1, \ldots, m\}$ is non-empty and if $\sum_{i=1}^{m} v_i a_{ij} \geqq \gamma_j$ for all $j = 1, \ldots, n$, then a solution of the game G exists.*

Proof. To the decomposition $\{(1, \ldots, m), \emptyset, \emptyset\}$ there corresponds the row in Q with non-negative components $-\gamma_j + \sum_{i=1}^{m} v_i a_{ij}$, $j = 1, \ldots, n$.

[1]) The theorem holds under the assumption of convexity (concavity) only; the existence of the second derivatives is not needed.

6. If the assumption $\mathsf{E}(b_i) = \beta_i$ is replaced by $\bar{\beta}_i \leqq \mathsf{E}(b_i) \leqq \bar{\bar{\beta}}_i$, $i = 1, \ldots, m$, and all the other assumption of Section 5 are unaltered, then the Theorem 2 remains true with β_i replaced by $\bar{\beta}_i$ (in the definitions of λ_i and k_ν).

On the other hand, if the assumption $\beta'_i \leqq b_i \leqq \beta''_i$ a.s., is replaced by $\sigma(b_i) \leqq \leqq$ const. (say σ_i), $i = 1, \ldots, m$, and all the other assumptions of Section 5 are unaltered, then part (i) of the Theorem 2 remains true — this follows from Loève [5, Theor. 11.4.A,B] — but to find a solution is much more difficult. It means first to find

$$\max_{F \in \mathscr{F}} \mathsf{E}_F\{(\sum_{j=1}^{n} a_{ij}x_j - b_i)^+\} = \min\{d_0 + d_1\beta_i + d_2(\sigma_i^2 + \beta_i^2)\},$$

where the minimum is to be taken over all (d_0, d_1, d_2) which satisfy $(\sum_{j=1}^{n} a_{ij}x_j - y)^+ \leqq \leqq d_0 + d_1 y + d_2 y^2$, $-\infty < y < +\infty$, when $x \in E_n^+$ is fixed (see Richter [6]), then to set this result into the relation for $\varphi(x)$ and to maximize $\varphi(x)$. (Both these comments are easily to prove.)

7. In the game G, let $(a_{i1}, \ldots, a_{in}, b_i)$, $i = 1, \ldots, m$ be mutually independent random vectors such that

$$\alpha'_{ij} \leqq a_{ij} \leqq \alpha''_{ij},\ \beta'_i \leqq b_i \leqq \beta''_i \ \text{a.s.},\ \alpha'_{ij} < \alpha''_{ij}, \tag{4}$$

$$\mathsf{E}a_{ij} = \alpha_{ij} = \tfrac{1}{2}(\alpha'_{ij} + \alpha''_{ij}),\quad \mathsf{E}b_i = \beta_i = \tfrac{1}{2}(\beta'_i + \beta''_i),\quad i = 1, \ldots, m,\quad j = 1, \ldots, n;$$

let c_j, $j = 1, \ldots, n$ be random variables such that $\mathsf{E}c_j = \gamma_j$. Let $r_i(y) = v_i y^+$, $v_i > 0$, $i = 1, \ldots, m$. Let $\Delta_\nu = (I_\nu, J_\nu, K_\nu)$, $\nu = 1, \ldots, M$, be all those decompositions of $\{1, \ldots, m\}$, for which the sets

$$\{x \geqq 0 : \sum_{j=1}^{n} \alpha'_{ij}x_j - \beta''_i \geqq 0,\ i \in I_\nu;$$

$$\sum_{j=1}^{n} \alpha'_{ij}x_j - \beta''_i \leqq 0 \leqq \sum_{j=1}^{n} \alpha''_{ij}x_j - \beta'_i,\ i \in J_\nu;\ \sum_{j=1}^{n} \alpha''_{ij}x_j - \beta'_i \leqq 0,\ i \in K_\nu\}$$

are non-empty. Define

$$p_{\nu j} = -\gamma_j + \sum_{i \in I_\nu} v_i\alpha_{ij} + \tfrac{1}{2}\sum_{i \in J_\nu} v_i\alpha''_{ij},$$

$$h_\nu = \sum_{i \in I_\nu} v_i\beta_i + \tfrac{1}{2}\sum_{i \in J_\nu} v_i\beta'_i$$

and denote $P = (p_{\nu j})$, $\nu = 1, \ldots, M$, $j = 1, \ldots, n$; $h = (h_\nu)$, $\nu = 1, \ldots, M$.

Theorem 4. (i) *Relation* (1) *holds.* (ii) x *is a solution of the game* G *if and only if* (x, y_0) *is a solution of the linear program*

$$Px + ye \leqq h, \quad x \geqq 0, \quad y = \text{maximum}.$$

(Note that the case $\beta'_i = \beta''_i$ is not excluded.)

Proof. According to the theorem of Jiřina - Nedoma [4]

$$\max_{F\in\mathcal{F}} \mathsf{E}_F\{(\sum_{j=1}^n a_{ij}x_j - b_i)^+\} =$$

$$= \frac{(\sum_{j=1}^n \alpha''_{ij}x_j - \beta'_i) - (\sum_{j=1}^n \alpha_{ij}x_j - \beta_i)}{(\sum_{j=1}^n \alpha''_{ij}x_j - \beta'_i) - (\sum_{j=1}^n \alpha'_{ij}x_j - \beta''_i)} (\sum_{j=1}^n \alpha'_{ij}x_j - \beta''_i)^+ +$$

$$+ \frac{(\sum_{j=1}^n \alpha_{ij}x_j - \beta_i) - (\sum_{j=1}^n \alpha'_{ij}x_j - \beta''_i)}{(\sum_{j=1}^n \alpha''_{ij}x_j - \beta'_i) - (\sum_{j=1}^n \alpha'_{ij}x_j - \beta''_i)} (\sum_{j=1}^n \alpha''_{ij}x_j - \beta'_i)^+ =$$

$$= \tfrac{1}{2}[(\sum_{j=1}^n \alpha'_{ij}x_j - \beta''_i)^+ + (\sum_{j=1}^n \alpha''_{ij}x_j - \beta'_i)^+] .$$

The other parts of the proof are the same as in Theorem 2.

Condition (4) is essential for the linearity of resulting program only. Theorem 3 and its Corollary hold also true with P in the place of Q.

8. In the game G, let (A, b, c) be a random vector such that

$$\alpha'_{ij} \leqq a_{ij} \leqq \alpha''_{ij},\ \beta'_i \leqq b_i \leqq \beta''_i,\ \gamma'_j \leqq c_j \leqq \gamma''_j \quad \text{a.s.,} \quad j = 1, \ldots, n,\ i = 1, \ldots, m .$$

Let $r_i(y) = v_i y^+$, $v_i > 0$, $i = 1, \ldots, m$. Let $\Delta'_v = (I_v, J_v)$, $v = 1, \ldots, M'$, be all those decompositions of $\{1, \ldots, m\}$, for which the sets

$$\{x \geqq 0 : \sum_{j=1}^n \alpha''_{ij}x_j - \beta'_i \geqq 0,\ i \in I_v;\ \sum_{j=1}^n \alpha''_{ij}x_j - \beta'_i \leqq 0,\ i \in J_v\}$$

are non-empty. ($M' \leqq 2^m$ holds.) Define

$$r_{vj} = -\gamma'_j + \sum_{i\in I_v} v_i\alpha''_{ij}, \quad l_v = \sum_{i\in I_v} v_i\beta'_i$$

and denote $R = (r_{vj})$, $v = 1, \ldots, M'$, $j = 1, \ldots, n$; $l = (l_v)$, $v = 1, \ldots, M'$.

Theorem 5. (i) *Relation* (1) *holds.* (ii) *x is a solution of the game G if and only if (x, y_0) is a solution of the linear program*

$$Rx + ye \leqq l , \quad x \geqq 0 , \quad y = \text{maximum} .$$

Proof. Assertion (i) follows from the fact that condition (i) of Theorem 1 is satisfied. To prove assertion (ii), let us take into account that for $x \in E_n^+$ we have

$$\min_{F\in\mathcal{F}} \mathsf{E}_F\{\sum_{j=1}^n c_jx_j - \sum_{i=1}^m v_i(\sum_{j=1}^n a_{ij}x_j - b_i)^+\} = \sum_{j=1}^n \gamma'_jx_j - \sum_{i=1}^m v_i(\sum_{j=1}^n \alpha''_{ij}x_j - \beta'_i)^+ .$$

The rest of the proof is the same as in Theorem 2.

Theorem 3 and its Corollary also hold true with R in the place of Q.

9. It is possible to modify the above results for the case where $r_i(y) = 0$ for $y = 0$ and $r_i(y) > 0$ for $y \neq 0$, which corresponds to the relation $Ax = b$ at the place of $Ax \leqq b$ in the initial linear program. Then Theorem 1 remains true; if $r_i(y) = v_i y^+ + w_i y^-$, $v_i > 0$, $w_i > 0$, $i = 1, \ldots, m$, and all the other assumptions are unaltered, then Theorem 2 holds true with $\tilde{Q}$ and $\tilde{k}$ in the places of Q and k, where

$$\tilde{q}_{\nu j} = -\gamma_j + \sum_{i \in I_\nu} v_i a_{ij} + \sum_{i \in J_\nu} (v_i \lambda_i - w_i(1 - \lambda_i))\, a_{ij} - \sum_{i \in K_\nu} w_i a_{ij}\,,$$

$$\tilde{k}_\nu = \sum_{i \in I_\nu} v_i \beta_i + \sum_{i \in J_\nu} (v_i \lambda_i \beta'_i - w_i(1 - \lambda_i)\, \beta''_i) - \sum_{i \in K_\nu} w_i \beta_i\,,$$

and so on.

References

[1] *Blackwell D.* and *Girshick M. A.*: Theory of games and statistical decisions, 1954.

[2] *Fan Ky:* Minimax theorems, Proc. Nat. Acad. Sci. USA, *39* (1953), p. 42–47.

[3] *Iosifescu M.* and *Theodorescu R.*: Sur la programmation linéaire, C. R. Acad. Sci. Paris, *256* (1963), p. 4831–4833.

[4] *Jiřina M.* and *Nedoma J.*: Minimaximální řešení výběrového přejímacího postupu, Apl. mat., *1* (1956), p. 296–314.

[5] *Loève M.*: Probability theory, 1955.

[6] *Richter H.*: Parameterfreie Abschätzung und Realisierung von Erwartungswerten, Bl. Dtsch. Ges. Versicherungsmath., *3* (1957), p. 147–162.

Author's address: Praha 8 - Karlín, Sokolovská 83 (Matematicko-fyzikální fakulta KU).

Resumé

O MINIMAXOVÉM ŘEŠENÍ ÚLOHY STOCHASTICKÉHO LINEÁRNÍHO PROGRAMOVÁNÍ

JITKA ŽÁČKOVÁ, Praha

Obsahem článku je vyšetřování úlohy stochastického lineárního programování a jejího řešení jakožto strategické hry. V prvých paragrafech je tato úloha formulována a pro uvažovanou hru je dokázána věta o minimaxu (věta 1). V dalších paragrafech se studují metody řešení. Ve větách 2, 4 a 5 je dokázáno, že minimaxové řešení úlohy stochastického lineárního programování je v některých důležitých speciálních případech ekvivalentní řešení úlohy lineárního programování (větší dimense). Podmínky pro existenci konečného řešení jsou uvedeny ve větě 3.

Резюме

О МИНИМАКСНОМ РЕШЕНИИ ПРОБЛЕМЫ СТОХАСТИЧЕСКОГО ЛИНЕЙНОГО ПРОГРАММИРОВАНИЯ

ЙИТКА ЖАЧКОВА (Jitka Žáčková), Прага

В настоящей статье изучается задача стохастического линейного программирования и ее решение с точки зрения теории игр. В первых отделах дана постановка задачи, и для возникшей бесконечной игры показана теорема о минимаксе (теорема 1). Остальные отделы посвящены исследованию решения задачи. Для некоторых важных частных случаев показано (теоремы 2, 4 и 5), что минимаксное решение задачи стохастического линейного программирования равносильно решению определенной задачи линейного программирования (большей размерности). В теореме 3 исследуется существование конечного решения.