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Applications of stochastic programming: Achievements and questions

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Abstract

When solving a decision problem under uncertainty via stochastic programming it is essential to choose or to build a suitable stochastic programming model taking into account the nature of the real-life problem, character of input data, availability of software and computer technology. Besides a brief review of history and achievements of stochastic programming, selected modeling issues concerning applications of multistage stochastic programs with recourse (the choice of the horizon, stages, methods for generating scenario trees, etc.) will be discussed. © 2002 Published by Elsevier Science B.V.

Keywords: Stochastic programming; Multistage problems with recourse; Horizon; Stages

1. History, achievements and problems to be solved

Forty-five years ago, stochastic programming was set up independently by Beale [2], Dantzig [13], Charnes and Cooper [10] and others who observed that for many linear programs to be solved, the values of the presumably known coefficients were not available. They suggested to replace the deterministic view by a stochastic one assuming that these unknown coefficients or parameters are random and their probability distribution P is *known* and independent of the decision variables.

The prototype stochastic program (we focus on in the sequel) aims at the selection of the "best possible" decision which fulfills given "hard" constraints, say $\boldsymbol{x} \in \mathscr{X}$, accepting that the outcome of this decision is influenced by the realization of a random event ω. The realization of ω is not known at the time of decision, however, to get the decision one uses the knowledge of the probability distribution P of ω. The random outcome of a decision $\boldsymbol{x} \in \mathscr{X}$ is quantified as $f_0(\boldsymbol{x}, \omega)$. If the set of possible realizations of ω is finite, say $\{\omega^1, \dots, \omega^S\}$, methods of multiobjective programming suggest to choose a solution efficient with respect to the objective functions $f_0(\bullet, \omega^s),\ s = 1, \dots, S$. Such efficient solutions can be obtained, e.g., by minimization (or maximization) of a weighted sum of $f_0(\boldsymbol{x}, \omega^s),\ s = 1, \dots, S$. In our stochastic setting, the weights equal the known probabilities p^s of the atoms or scenarios ω^s of the probability distribution P and the problem to be solved is

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$$\min_{x \in \chi} \sum_{s=1}^{S} p^s f_0(\boldsymbol{x}, \omega^s).$$

Relaxation of hard constraints by the requirement that the constraints are fulfilled with a prescribed probability provides stochastic programs with *probabilistic* or *chance* constraints.

While simplifications cannot be avoided, multimodeling can help to discover model misspecifications. Complete knowledge of the underlying probability distribution cannot be expected and, in addition, approximations are needed to get numerically tractable problems. At the same time, optimal solutions of the approximate stochastic program should not be used without any further analysis in the place of the sought solution of the "true" problem.

The results of theoretical analysis and software development for various types of stochastic programming models were influenced and supported by developments in optimization, probability and statistics and in computer technologies, with the progress recorded step by step in textbooks and monographs [3,24,37,38,46], in surveys, e.g. [49], in special conference volumes, e.g. [14], published dissertations and in numerous focused issues of journals, see the preface of [50] for an extensive list of references. In the 1980s, a special care was devoted to software development and has resulted in the IIASA volume [25], in a recommended input format [5], in several monographs, e.g. [34,35,41], software packages and test batteries.

The first applications appeared already in the 1950s, e.g., [11,27]. They were based on simple types of stochastic programming models such as models with individual probabilistic constraints and stochastic linear programs with simple recourse. Moreover, special assumptions about the probability distribution P were exploited.

From the modeling point of view, stochastic vehicle routing, stochastic networks and stochastic facility locations problems have been mostly treated as a natural extension of the stochastic transportation problem with simple recourse, whereas individual probabilistic constraints have appeared in the context of the stochastic nutrition model and in water resources management models.

The significant progress in the 1970s [45] facilitated application of joint probabilistic constraints. An impressive early collection of case studies related to their real-life applications is [47]; it reflects the prevailing interest in applications to water resources problems at that time.

Another important extension, to multistage stochastic programs, has aimed at a more realistic treatment of the dynamic or sequential structure of real-life decision problems. Several essential contributions in this direction have appeared already in [14,47,51]. Sophisticated approaches to portfolio management from that time, e.g. [6], have become the cornerstone of the contemporary financial applications and have contributed also to modeling and software development for multistage stochastic programs.

At present, the most popular seem to be financial applications of stochastic programming. The list of further favorable application areas contains for instance planning and allocation of resources (including water), energy production and transmission, production planning and optimization of technological processes, logistics problems (including aircraft allocation and yield management), and telecommunications.

To summarize the achievements

- There are *standardized types of stochastic programming models* (e.g., two-stage and multistage stochastic programs with recourse, models with individual and joint probabilistic constraints, integer stochastic programs) with links to statistics and probability, to parametric and multiobjective programming, to stochastic dynamic programming and stochastic control with relevant *software systems* available or in progress.
- There exist successful *large-scale real-life applications*. It became clear that their success is conditioned by a close collaboration with the users and that one can benefit from team work. Areas of further prospective applications have been delineated.

- Also the *tradition* of the triennial International Conferences on Stochastic Programming and numerous workshops, existence of focused groups, such as the Committee of Stochastic Programming (COSP) established in 1981 within the Mathematical Programming Society, or the WG 7.7 of IFIP, the periodically updated stochastic programming bibliography [48] and the stochastic programming electronic preprint series (SPEPS) belong among the evident achievements of the field.

The present boom of large-scale real-life applications has brought new challenging questions. An important task is an adequate reflection of the dynamic aspects, including further development of tractable numerical approaches. Additional problems are related with the fact that the probability distribution P is rarely known completely and/or that it has to be approximated for reasons of numerical tractability so that one mostly solves an approximate stochastic program instead of the underlying true decision problem. The task is to generate the required input, i.e., to approximate P bearing in mind the required type of the problem; see e.g. [23]. Moreover, without additional analysis, the obtained output (the optimal value and optimal solutions of the approximate stochastic program) should not be used to replace the sought solution of the true problem; see [20,21] for discussion of suitable *output analysis methods*. These methods have to be tailored to the structure of the problem and they should also reflect the source, character and precision of the input data.

As in current software systems, the methods of output analysis address at present mainly the two-stage (multiperiod) stochastic programs. The reason is that the structure of multistage problems is much more involved and one cannot rely on intuitive straightforward generalizations. At the same time validation experiments, e.g. [52], provide an evidence that even three-stage stochastic programs may outperform significantly the existing static models. Hence, an extensive all-round research in multistage stochastic programming is an important complex task of the day.

2. Multistage stochastic programs

In the general *T-stage stochastic program* we think of a stochastic data process

$$\omega = (\omega_1, \ldots, \omega_{T-1}) \quad \text{or} \quad \omega = (\omega_1, \ldots, \omega_T)$$

whose realizations are (multidimensional) data trajectories and of a vector decision process

$$\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_T),$$

a measurable function of ω. The sequence of decisions and observations is

$$\boldsymbol{x}_1, \omega_1, \boldsymbol{x}_2(\boldsymbol{x}_1, \omega_1), \omega_2, \ldots, \boldsymbol{x}_T(\boldsymbol{x}_1, \omega_1, \ldots, \omega_{T-1}). \tag{1}$$

Realizations of ω_T, i.e., those behind the horizon, do not affect the decision process, they may contribute to the overall costs. The decision process may be affected by the *probability distribution* of ω_T. The decision process is *nonanticipative* in the sense that a sequence of decisions is built along each of the considered data trajectories in such a way that decisions based on the same part of trajectory, on the same history, are identical. It means that decisions taken at any stage of the process do not depend on future *realizations* of random parameters or on future decisions, whereas it is the past information and the knowledge of the probabilistic specification $(\Omega, \mathscr{F}, P)$ of the process ω which are exploited. The dependence of the decisions solely on the history and on the probability specification can be mathematically expressed as follows: denote $\mathscr{F}_{t-1} \subseteq \mathscr{F}$ the σ-field generated by the observations of $\omega^{t-1,\bullet} := (\omega_1, \ldots, \omega_{t-1})$, i.e., of the part of the stochastic data process that precedes stage t. The dependence of the tth stage decision $\boldsymbol{x}_t$ only on these past observations means that $\boldsymbol{x}_t$ is $\mathscr{F}_{t-1}$-adapted or, in other words, that $\boldsymbol{x}_t$ is measurable with respect to $\mathscr{F}_{t-1}$. In each of the stages, the decision is limited by constraints that may depend on the previous decisions and observations.

The first-stage decisions consist of all decisions that have to be selected before further information is revealed whereas the second-stage decisions are allowed to adapt to this information, etc. Stages do not necessarily refer to time periods, they correspond to steps in the decision process.

With reference to the survey paper [18] and references therein, we shall present here only two formulations frequently used for the multistage stochastic linear programs with recourse.

In the first formulation, the nested structure of the problems related to individual stages resembles the backward recursion common in *stochastic dynamic programming*:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}_1^\top \boldsymbol{x}_1 + E_{\omega_1}\{\varphi_1(\boldsymbol{x}_1, \omega_1)\} \\ \text{subject to} \quad & \boldsymbol{A}_1 \boldsymbol{x}_1 = \boldsymbol{b}_1, \\ & \boldsymbol{l}_1 \leqslant \boldsymbol{x}_1 \leqslant \boldsymbol{u}_1, \end{aligned} \tag{2}$$

where the functions $\varphi_{t-1},\ t = 2, \ldots, T$, are defined recursively as

$$\begin{aligned} & \varphi_{t-1}(\boldsymbol{x}_{t-1}, \omega_{t-1}) = \inf_{\boldsymbol{x}_t} \left[\boldsymbol{c}_t(\omega_{t-1})^\top \boldsymbol{x}_t + E_{\omega_t|\omega_{t-1}}\{\varphi_t(\boldsymbol{x}_t, \omega_t)\} \right] \\ \text{subject to} \quad & \boldsymbol{B}_t(\omega_{t-1})\boldsymbol{x}_{t-1} + \boldsymbol{A}_t(\omega_{t-1})\boldsymbol{x}_t = \boldsymbol{b}_t(\omega_{t-1}), \quad \text{a.s.}, \\ & \boldsymbol{l}_t(\omega_{t-1}) \leqslant \boldsymbol{x}_t \leqslant \boldsymbol{u}_t(\omega_{t-1}) \end{aligned} \tag{3}$$

and $\varphi_T \equiv 0$ or it is an explicitly given function of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_T, \omega_1, \ldots, \omega_T$ if contribution of ω_T is taken into account. Constraints involving random parameters hold almost surely.

For simplicity, we denote by ω_{t-1} the random vector that generates the vectors of coefficients $\boldsymbol{b}_t, \boldsymbol{c}_t, \boldsymbol{l}_t, \boldsymbol{u}_t$ and matrices $\boldsymbol{A}_t, \boldsymbol{B}_t$ in the tth stage decision problem (3), $t = 2, \ldots, T$. We assume that $\boldsymbol{A}_t$ are (m_t, n_t) matrices and that the remaining vectors and matrices are of consistent dimensions. The Markov structure of constraints and of the objective in (3) is not essential. We suppose, however, that the corresponding expectations E are well defined. For the first stage, known values of all elements of $\boldsymbol{b}_1, \boldsymbol{c}_1, \boldsymbol{A}_1, \boldsymbol{l}_1, \boldsymbol{u}_1$ are assumed. The main decision variable is $\boldsymbol{x}_1$ that corresponds to the first stage. It is relatively easy to prove that, under the mentioned assumptions, the first-stage problem (2) is a convex program.

In spite of the formal similarity with the stochastic dynamic programming problems, this form does not enter the numerical procedures for solving stochastic programs. The main interest lies in the first-stage decisions. Even if it is often possible to characterize the decision rules, it is not necessary to design a full backward recursion as in dynamic programming and, due to large dimensionality of stochastic programming problems, such procedure would be hardly tractable. The dynamic decision process is approximated by optimal solutions obtained by repeated solution of similar stochastic programs which are *rolled forward* in time, i.e., by solving the problem repeatedly starting always with the new state of the system attained by application of the obtained optimal first-stage decision and using updated and/or shifted data trajectories.

For purposes of applications one mostly approximates the true probability distribution P of ω by a discrete probability distribution concentrated on a finite number of atoms $\omega^1, \ldots, \omega^S$. Accordingly, the supports of conditional probability distributions of ω_t conditioned by past realizations of $\omega_1, \ldots \omega_{t-1}$ and the supports of marginal probability distributions of the components $\omega_t\ \forall t$ are finite sets. The associated conditional probabilities are called the *arc* probabilities. A special common arrangement of the data process is the *scenario tree* which is based on the requirement that there is a one-to-one correspondence between the history $\omega^{t-1,\bullet} = (\omega_1, \ldots, \omega_{t-1})$ and one of the nodes (states of the system) at the stage t. The corresponding "arborescent" form of the *T-stage scenario-based stochastic linear program with recourse* reads:

$$\text{minimize} \quad \boldsymbol{c}_1^\top \boldsymbol{x}_1 + \sum_{k_2=2}^{K_2} p_{k_2} \boldsymbol{c}_{k_2}^\top \boldsymbol{x}_{k_2} + \sum_{k_3=K_2+1}^{K_3} p_{k_3} \boldsymbol{c}_{k_3}^\top \boldsymbol{x}_{k_3} + \cdots + \sum_{k_T=K_{T-1}+1}^{K_T} p_{k_T} \boldsymbol{c}_{k_T}^\top \boldsymbol{x}_{k_T} \tag{4}$$

$$\begin{array}{llllll} \text{subject to} & \boldsymbol{A}_1\boldsymbol{x}_1 & & & = \boldsymbol{b}_1, & \\ & \boldsymbol{B}_{k_2}\boldsymbol{x}_1 + & \boldsymbol{A}_{k_2}\boldsymbol{x}_{k_2} & & = \boldsymbol{b}_{k_2}, & k_2 = 2,\ldots,K_2, \\ & & \boldsymbol{B}_{k_3}\boldsymbol{x}_{a(k_3)} & +\boldsymbol{A}_{k_3}\boldsymbol{x}_{k_3} & = \boldsymbol{b}_{k_3}, & k_3 = K_2+1,\ldots,K_3, \\ & & \ddots & \ddots & \vdots & \\ & & & \boldsymbol{B}_{k_T}\boldsymbol{x}_{a(k_T)} \quad +\boldsymbol{A}_{k_T}\boldsymbol{x}_{k_T} & = \boldsymbol{b}_{k_T}, & k_T = K_{T-1}+1,\ldots,K_T. \end{array} \tag{5}$$

$$\boldsymbol{l}_{k_t} \leqslant \boldsymbol{x}_{k_t} \leqslant \boldsymbol{u}_{k_t}, \quad k_t = K_{t-1}+1,\ldots,K_t, \quad t = 1,\ldots,T.$$

We denote here by $a(k_t)$ the immediate ancestor of k_t, so that (with $K_1 = 1$) $a(k_2) = 1,\ k_2 = 2,\ldots,K_2$. The problem is based on the used scenarios, i.e., on the $S = K_T - K_{T-1}$ sequences of possible realizations $(\boldsymbol{c}_{k_t}, \boldsymbol{A}_{k_t}, \boldsymbol{B}_{k_t}, \boldsymbol{b}_{k_t}, \boldsymbol{l}_{k_t}, \boldsymbol{u}_{k_t})$ of coefficients in the objective function (4), in recourse matrices, transition matrices and right-hand sides in the constraints for all stages, and on *path probabilities* $p_{k_t} > 0\ \forall k_t$, $\sum_{k_t=K_{t-1}+1}^{K_t} p_{k_t} = 1,\ t = 2,\ldots,T$, of partial sequences of these coefficients that identify the discrete distribution P. The path probabilities are obtained by multiplication of the (conditional) *arc probabilities* of the corresponding sequences of realizations. The probabilities p^s of the individual scenarios ω^s are equal to path probabilities $p_{K_t},\ k_t = K_{T-1}+1,\ldots,K_T$.

The nonanticipativity constraints are included in an implicit form. Decomposition of (4) and (5) along scenarios is possible but it requires that the nonanticipativity constraints are spelled out in an explicit way.

The size of the linear program (4) and (5) can be very large and usefulness of special numerical techniques is obvious. Still, in real-life applications, it is the modeling part of the problem and a meaningful generation of scenarios which have become the most demanding task.

Besides the formulation of goals and constraints and identification of the driving random process ω, building a scenario-based multistage stochastic program requires specification of the horizon, stages and generation of the input in the form of scenario tree.

2.1. An illustrative example

The flower-girl problem introduced in [7] is a simple multistage stochastic program. The flower girl sells roses at c and has to buy them at p before she starts selling. Flowers left over at the end of the day can be stored and sold the next day, when she starts selling the old roses. The roses cannot be carried over more than one additional day at the end of which they are thrown away. The demand is random, ω_t denotes the demand on the tth day. The flower girl wants to maximize her expected profit.

The horizon is related to the number of days for which the flower girl continues selling roses without any break (and also to the fact that our formulation treats only one-period carryover). Assume first that the flower girl sells roses only during the weekend, orders the amount x_1 on Friday evening, observes the demand ω_1 on Saturday, stores the unsold roses (without any additional cost) and, possibly, buys $x_2(\omega_1)$ new roses. Denote $s_1(\omega_1)$ the stock left for the subsequent day and $z_2(\omega_1, \omega_2)$ the amount of unsold roses at the end of the second day.

All decision variables are nonnegative and subject to constraints

$$x_1 - s_1(\omega_1) \leqslant \omega_1,$$

$$x_2(\omega_1) + s_1(\omega_1) - z_2(\omega_1, \omega_2) \leqslant \omega_2.$$

If the demand ω_1, ω_2 is known in advance, the objective function is

$$(c-p)(x_1 + x_2(\omega_1)) - cz_2(\omega_1, \omega_2)$$

and one of the optimal solutions is to buy $x_1 = \omega_1$ and $x_2 = \omega_2$ roses which gives the maximal profit of $(c-p)(\omega_1 + \omega_2)$. Consider now a scenario-based version of this three-stage problem. The scenario tree consists of K branches corresponding to the considered realizations $\omega_1^k,\ k = 1, \ldots, K$, of the demand on the first day, their probabilities are $p_k,\ k = 1, \ldots, K$. Possible realizations of demand $\omega_2^{k\sigma}$ for the second day may be conditional on ω_1^k. We denote $\mathscr{D}(\omega_1^k)$ the set of descendants of ω_1^k, and $\pi_{k\sigma}$ their (conditional) probabilities. The problem is

$$\text{maximize} \quad (c-p)x_1 + \sum_{k=1}^{K} p_k \left[(c-p)x_2(\omega_1^k) - c \sum_{\sigma \in \mathscr{D}(\omega_1^k)} \pi_{k\sigma} z_2(\omega_1^k, \omega_2^{k\sigma}) \right]$$

$$\text{subject to:} \quad x_1 - s_1(\omega_1^k) \leqslant \omega_1^k, \quad k = 1, \ldots, K,$$

$$x_2(\omega_1^k) + s_1(\omega_1^k) - z_2(\omega_1^k, \omega_2^{k\sigma}) \leqslant \omega_2^{k\sigma}, \quad \sigma \in \mathscr{D}(\omega_1^k), \quad k = 1, \ldots, K,$$

and nonnegativity constraints. The total number S of scenarios $(\omega_1^k, \omega_2^{k\sigma})$ equals the number of all descendants of $\omega_1^k,\ k = 1, \ldots, K$.

The generalization to $(T+1)$-stage problem is obvious:

$$\text{maximize} \quad (c-p)x_1 + E\left\{ (c-p)\sum_{t=2}^{T} x_t(\omega^{t-1,\cdot}) - c\sum_{t=1}^{T} z_t(\omega^{t-1,\cdot}, \omega_t) \right\}$$

$$\text{subject to:} \quad x_1 + s_0 - s_1(\omega_1) - z_1(\omega_1) \leqslant \omega_1,$$

$$x_t(\omega^{t-1,\cdot}) + s_{t-1}(\omega^{t-1,\cdot}) - s_t(\omega^{t,\cdot}) - z_t(\omega^{t-1,\cdot}, \omega_t) \leqslant \omega_t, \quad t = 2, \ldots, T,$$

$$s_{t-1}(\omega^{t-1,\cdot}) - z_t(\omega^{t-1,\cdot}, \omega_t) \leqslant \omega_t, \quad t = 1, \ldots, T,$$

with $s_T(\omega) \equiv 0$ and nonnegativity of all variables. In case the initial supply $s_0 = 0$, one gets $z_1(\omega_1) \equiv 0$. The number of stages equals one plus the number of days for which the flower girl sells roses without any break. The scenario-based formulation of the T-stage problem can be written in the arborescent form or in the scenario splitted form with explicit nonanticipativity constraints.

Imagine now that the flower girl wants to earn as much as possible during the two months of her high school vacations; such a 63-stage problem may be solvable thanks to its simple form. Still some other possibilities should be examined. Her problem may be rolled forward in time with a substantially shorter horizon, say, with $T = 8$ which covers a whole week. This means that the flower girl decides as if she plans to maximize her profit over each one-week period and solves the problem every day with a known (possibly nonzero) initial supply of roses and with a new scenario tree spanning over the next eight days. Another possibility is the aggregation of stages. With a long horizon and random parameters only on the right-hand side of the constraints, one may apply the idea of [32] designed for problems with an infinite horizon: one chooses a tractable horizon T and adds one stage which takes into account the remaining stages $t \geqslant T$.

In a majority of papers, the horizon and stages are declared as given. In practice, various situations can be distinguished:

- Both the horizon and stages are determined ad hoc, often for purposes of testing numerical approaches and/or software both with or without rolling horizon simulations.
- Both the horizon and stages are determined, e.g., by the nature of the real-life technological process [44]; another example is the flower-girl problem.
- The horizon is tied to a fixed date, e.g., to the end of the fiscal or hydrological year, to a date related with the annual Board of Directors' meeting, or to the end date of a screening study. Stages are sometimes dictated by the nature of the solved problem, e.g., by the dates of maturity of bonds [29] or expiration dates of options or by periodic (quarterly, annual, etc.) management review meetings. In other cases, they are obtained by application of heuristic rules and/or experience, taking into account limitations due to numerical tractability. We refer to the discussion in [43] for financial applications, to [36,42] for scheduling hydroelectric generation, to [31] for harvest optimization with horizon of 120 years and to [4] for an investment planning study over a horizon of more than 100 years. Rolling forward after the T-stage

problem has been solved leads to a subsequent $(T-1)$-stage stochastic program with a reduced number of stages or possibly to another T-stage problem with a different topology of stages, cf. [36].
- The horizon is connected with a time interval of a fixed (possibly even infinite) length, given for instance by the periodicity of the underlying random process, and the number of stages is chosen as a function of the available computing facilities. Rolling forward means here the repeated solution of a T-stage problem with the same structure of stages, with the initial state of the system determined by the applied first-stage decision and by the observed value of ω_1, and using the process ω shifted forward in time.

For example, [6] uses three one-year periods for the three-year planning horizon of the bank and rolling forward means that each year the bank is planning as if it wants to optimize its outcome at the end of the next three years. For production planning problems, [26] suggests a horizon of 12–18 months divided into three stages. Energy generation models, such as [33], are usually built for one-week horizon subdivided into 2–12 stages. Short term hydropower system control may use a horizon of 3 hours subdivided more or less arbitrarily into stages [35], whereas the "long" term planning can be related to a weekend. In the last case, rolling forward will have the same meaning as for the fixed horizon problems. Infinite horizon models are approximated by those with a finite horizon; see [32] whose suggestion – to aggregate the future in a stationary stage – was applied for example in [8].

There exist further specific features of the solved problems. For instance, the problem can be solved just once (to retire the debt by a given deadline as much as possible in [15]) or the problem and its solution persist in the future, with new horizons, taking always into account just the final state of the system at the previous termination date, i.e., at the previous horizon. To guarantee the possibility of such continuation, the models are usually extended for additional constraints and/or terms in the objective function to reduce the end effects, with or without reference to an additional, auxiliary stage.

For a chosen horizon, the crucial step is to relate the time instants and stages; this is a common problem both in applications of multistage stochastic programming models and in stochastic dynamic programming with discrete time. Some recommendations are common for financial applications [8,43]: Accept unequal lengths of time periods between subsequent stages, starting with a short first period. Together with repeated rolling of the model over time, this may replace well the full dynamics of the decision process even for problems with a few stages. Another, general suggestion [32] is to break the problem with a long (possibly infinite) horizon into three phases: To use the scenario tree structure for $1 \leqslant t \leqslant T$, to design just one descendant from each node for $T+1 \leqslant t \leqslant \tau$ (i.e., the horse-tail structure) and to aggregate the rest of the process into one additional stationary stage. Moreover, in reality, the position of stages can be uncertain, random or scenario dependent – an interesting open problem. The main limitations of the number of stages are due to numerical tractability.

The basic information on modeling horizon and stages in selected financial applications is contained in Table 1; in the last column, the acronym R1 is used for rolling forward with a fixed horizon, R2 with a shorter horizon, YFF for end effect treatment according to [32].

There are only a few papers which compare the optimal first-stage decisions in dependence on the *number of stages*, e.g., [16,31,40].

Starting with a given initial structure of the problem, one generates the input accordingly. This includes designing the *branching scheme* of the scenario tree, cf. [23] and references therein. Evidently, one has to accept compromises between the size of the resulting problem and the desired precision of the results. A detailed analysis of the origin and of the initial structure of the solved problem may be exploited to aggregate the stages, may help to prune the tree or to extend it for other out-of-sample scenarios or branches. It is even possible to test the influence of including additional stages, e.g., using the contamination approach; see for instance [19,20,22] and further papers in [53].

To avoid the necessity of generating a scenario tree, one may try to aggregate stages and to reduce the solved problem into a two-stage model [40], or to design a battery of sequences of dynamic decisions and

Table 1
Structure of multistage SP models applied in finance

Ref.	Problem	T	Stages	Comparisons	End eff.
[1]	Pension company	30Y	INI, 2Y, 3Y, 5Y, 2×10Y	Various rules	
[6]	Bond portfolio	3Y	Yearly	Ladder, barbell	R1
[8]	ALM – insurance	5Y	INI, end 1Q, 1Y, 2Y, 5Y	Markowitz	YFF, R1
[9]	ALM	5Y	1Y, 2Y, 2Y	Fixed-mix	Bounds
[12]	Pension plan	10Y	Yearly		
[15]	Debt financing	5Y	4×1Q, 4×1Y		
[17]	Pension plan	10Y	Yearly	Static, myopic	Bounds
[28]	ALM – insurance	3Y	Yearly	Fixed-mix	
[39]	Pension fund	5Y	Yearly	Fixed-mix	R2
[52]	Financial product	3Y	INI, 18M, 36M	One-stage	

test them along data trajectories. Also the space of feasible solutions may be reduced by prescribing specific rule-based policies; a typical example is the fixed-mix policy in portfolio management, cf. [28,30,39].

3. Conclusions

There are many excellent recent papers on successful real-life applications of stochastic programming; it is impossible to list them all. We have referred to [53] for a selection of financial problems and there are ongoing projects of special collections devoted to nonfinancial applications. Nevertheless, these applications are rather demanding and there are still many open questions. Those discussed in this paper dealt with selected issues related with building multistage stochastic programming models with recourse. Evidently, before formulating further guidelines for specific application areas, more experience on the impact of the chosen horizon and/or of its discretization into stages on the results has to be collected using simulations, comparisons and output validation. Stochastic programs with probabilistic constraints or integer stochastic programs were not discussed here; their applications bring along further model and problem specific tasks.

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