STOCHASTIC PROGRAMMING: MINIMAX APPROACH

In many applications of stochastic programming there is some uncertainty about the probability distribution P of the random parameters. The *incomplete knowledge of the probability distribution* can be described by assuming that P belongs to a specified class $\mathcal{P}$ of probability distributions. This in turn suggests to use the *minimax decision rule*.

The first results were concerned with stochastic linear programs with recourse; they can be treated within the following more general framework

$$\text{minimize} \quad F(\mathbf{x}; P) := E_P f(\mathbf{x}; \omega) \quad \text{on the set } \mathcal{X} \subset R^n \tag{1}$$

with $\mathcal{X}$ a given set of decisions, P a probability distribution on $(\Omega, \Sigma), \Omega \subset R^m$ and P known to belong to a class $\mathcal{P}$. The random outcome of a decision $\mathbf{x} \in \mathcal{X}$ is quantified by a function f defined on $\mathcal{X} \times \Omega$, E_P denotes the expectation under P.

These results were formulated in terms of the *two-person zero-sum game*

$$(\mathcal{X}, \mathcal{P}, F(\mathbf{x}; P)). \tag{2}$$

Iosifescu and Theodorescu [11] suggested to use an optimal *mixed* strategy of the first player in the game (2). Žáčková [18] introduced the notion of *minimax solution as an optimal pure strategy of the first player in the game* (2). Under quite general assumptions on $\mathcal{P}$ and F, a minimax solution exists and

$$\inf_{\mathbf{x} \in \mathcal{X}} \max_{P \in \mathcal{P}} F(\mathbf{x}; P) = \max_{P \in \mathcal{P}} \inf_{\mathbf{x} \in \mathcal{X}} F(\mathbf{x}; P). \tag{3}$$

The minimax decision rule can be applied also in cases when the minimax theorem for the game (2) does not hold true. It means to solve the problem

$$\text{minimize} \quad \max_{P \in \mathcal{P}} F(\mathbf{x}; P) \quad \text{on the set} \quad \mathcal{X} \subset R^n \tag{4}$$

hence, to apply the best possible decision obtained for the most adverse considered circumstances. This provides a tool for the *worst case analysis* for program (1) and allows for *constructing bounds* for optimal value of (1) valid for all $P \in \mathcal{P}$.

Applicability of the results depends on the assumed form of the class $\mathcal{P}$ which describes the level of the available information about the probability distribution of the random parameters and also on the properties of the random objective function $f(\mathbf{x};\omega)$. Let us list some of the most frequent *choices of* $\mathcal{P}$:

- $\mathcal{P}$ consists of probability distributions carried by $\Omega \subset R^m$ which fulfil certain *moment conditions*, e. g.,

$$\mathcal{P} = \{P : E_P g_j(\omega) = y_j, j = 1, \ldots, J\} \tag{5}$$

  with prescribed values $y_j \, \forall j$ ([3], [6], [8], [17], [18]).

- $\mathcal{P}$ contains probability distributions on (Ω, Σ) with fixed marginals ([15]).

- An additional qualitative information, such as unimodality of P, is taken into account ([6], [8]).

- $\mathcal{P}$ consists of probability distributions P carried by a known *finite* support of P, i.e., to specify P means to fix the probabilities of the considered atoms (scenarios) taking into account a prior knowledge about their partial ordering, etc.; see e. g. [4].

- $\mathcal{P}$ is a neighborhood of a hypothetical probability distribution P_0 ([3]).

- In principle, $\mathcal{P}$ can be also a parametric family of probability distributions with an incomplete knowledge of parameter values.

For convex, compact $\mathcal{P}$, the expectation $F(\mathbf{x};P) = E_P f(\mathbf{x};\omega)$ attains its maximal (and minimal) value at extremal points of $\mathcal{P}$; the extremal probability distributions can be characterized independently of the form of the random objective f, however, the worst case probability distribution, say, $P^* \in \mathcal{P}$ independent of f (and thus independent of the decisions $\mathbf{x}$) appears only exceptionally. If this is possible the objective function in (4) $\max_{P\in\mathcal{P}} F(\mathbf{x};P) = F(\mathbf{x};P^*)$ is just an objective function of a standard stochastic program which is relatively easy to solve due to a relatively simple structure of P^*. There are also instances when one can succeed to get the explicit form of $\max_{P\in\mathcal{P}} F(\mathbf{x};P)$ ([6], [12]). They relate to classes of one-dimensional probability distribution and to special functions f.

The general methodology for solution of the inner optimization problem $\max_{P\in\mathcal{P}} F(\mathbf{x};P)$ for a fixed decision $\mathbf{x}$ has been elaborated in detail for the classes of probability distributions defined by moment conditions (5), both in the form of equations and inequalities: The extremal probability distributions have finite supports, cf. [14], [17], and the solution of the inner problem

$$\max_P \int_\Omega f(\mathbf{x};\mathbf{z})dP \tag{6}$$

$$\text{subject to } \int_\Omega dP = 1, \quad \int_\Omega g_j(\mathbf{z})dP = y_j, \quad j = 1,\dots,J \tag{7}$$

reduces to solution of a generalized linear program (cf. [2], [3], [7], [9], [17]) provided that Ω is compact and $f(\mathbf{x};\bullet), g_j, \forall j$ are continuous on Ω. The procedure provides both the atoms of the sought worst case probability distribution and their probabilities. In some cases, it is expedient to analyze the dual program to (6), (7) which reads

$$\min_{\mathbf{u}} \sum_{j=1}^J u_j y_j + u_0 \tag{8}$$

$$\text{subject to } u_0 + \sum_{j=1}^J u_j g_j(\mathbf{z}) \geq f(\mathbf{x};\mathbf{z}) \quad \forall \mathbf{z} \in \Omega. \tag{9}$$

For details and various applications consult [2], [3], [5]–[8], [9], [13], [17].

As an *example* consider $f(\mathbf{x};\bullet)$ a *convex* function on a bounded convex polyhedron $\Omega \subset R^m$, say, $\Omega = \text{conv}\{\omega^{(1)},\dots,\omega^{(K)}\}$ and

$$\mathcal{P} = \{P : E_P\omega_j = y_j, j = 1,\dots,m\} \tag{10}$$

with $\mathbf{y}$ a given interior point of Ω. The constraints of (9)

$$u_0 + \sum_{j=1}^m u_j z_j \geq f(\mathbf{x};\mathbf{z})$$

hold true for all $\mathbf{z} \in \Omega$ if and only if they are fulfilled for the extremal points $\omega^{(1)},\dots,\omega^{(K)}$. Duality properties imply that only suitable subsets of the set of extremal points of Ω need to be considered in construction the finite supports of the worst case distributions. The generalized linear program (6), (7) reduces to the *linear* program

$$\max_{\mathbf{p}} \sum_{k=1}^K p_k f(\mathbf{x};\omega^{(k)}) \tag{11}$$

$$\text{subject to } \sum_{k=1}^{K} p_k \omega_j^{(k)} = y_j,\ j = 1, \ldots, m, \quad \sum_{k=1}^{K} p_k = 1,\ p_k \geq 0\ \forall k. \qquad (12)$$

Convexity of f with respect to ω is essential for the above result. Generalization to piecewise convex functions $f(\mathbf{x}, \bullet)$ (cf. [5]) is possible, on the other hand the worst case probability distribution from the class (10) for f *concave* in ω is the degenerated distribution concentrated at the prescribed expected value $E_P\omega$. This degenerated distribution provides the best (i.e., the minimal possible) expectation for convex functions $f(\mathbf{x}, \bullet)$ under P belonging to the class $\mathcal{P}$; compare with the Jensen inequality.

If the set of feasible solutions of (12) is a singleton the worst case distribution P^* does not depend on f and we obtain bounds for the optimal values of the stochastic programs (1) under an arbitrary probability distribution P from the class (10) and an arbitrary function f which is convex in ω:

$$\min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}, E_P\omega) \leq \min_{\mathbf{x} \in \mathcal{X}} E_P f(\mathbf{x}, \omega) \leq \min_{\mathbf{x} \in \mathcal{X}} E_{P^*} f(\mathbf{x}, \omega)\ \forall P \in \mathcal{P} \qquad (13)$$

provided that the minima exist. Such bounds are numerically tractable, are tight and provide an information about sensitivity of the optimal value of stochastic program (1) on the choice of a probability distribution P belonging to the considered class $\mathcal{P}$. The well known instance is the class of probability distributions carried by a closed interval $[a, b]$ on the real line with a prescribed value $y \in (a, b)$ of the expectation $E_P\omega$. The worst case distribution is carried by the end-points of the given interval $[a, b]$ and the only solution of the system

$$p_1 a + p_2 b = y,\ p_1 + p_2 = 1,\ p_1, p_2 \geq 0$$

is $p_1 = \frac{b-y}{b-a}$ and $p_2 = 1 - p_1$. The result agrees with the well-known Edmundson-Madansky inequality and the minimax approach guarantees that this bound is tight within the considered class of probability distributions and for convex functions $f(\mathbf{x}, \bullet)$.

There is a host of papers devoted to designing various *bounds* for the objective function $F(\mathbf{x}, P)$ of stochastic programs (1) under various assumptions about the class $\mathcal{P}$ and the function $f(\mathbf{x}, \bullet)$; for a review of the related results see [2], [3], [6], [13], [17] and references ibid. These bounds proved to be useful also in designing algorithms and this is at present the main field of successful applications of the minimax approach.

On the other hand, to get the *minimax decisions* is rather demanding as it requires solution of the full minimax problem (4). Except for the simple special cases, such as a unique feasible discrete distribution that fulfils (7) or the optimal value of the objective function (8) obtained in an explicit form, one has to rely on special numerical procedures such as the stochastic quasigradient methods designed for this purpose in [9], [10]. The numerical difficulties are behind the fact that, in spite of a sound motivation, real life applications of the minimax approach have been rare and have consisted of the simple special cases (e. g., [1], [4], [6], [8], [15], [16]).

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Keywords and phrases: stochastic programming, incomplete information, game theory, minimax decision rule, worst case analysis, bounds.
AMS1991 Subject Classification: 90C15, 62C20