STOCHASTIC GEOMETRIC PROGRAMMING: APPROACHES AND APPLICATIONS

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Abstract. During the last years, an increasing interest in geometric programming (GP) can be observed. Advances in numerical methods allow to solve large GPs and new areas of successful applications have emerged: besides of technical applications, there are also GPs for optimal production planning, finance, etc. In real-life applications of GP, some of coefficients and/or exponents need not be precisely known. Stochastic geometric programming can be used to deal with such situations. In this paper, we shall indicate which of general stochastic programming techniques and under which structural and distributional assumptions do not destroy the favorable structure of GPs. Both the already recognized and new approaches will be presented in connection with formulation of the optimization problem. The short note below should serve as an introduction to basic concepts and references.
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1 Geometric programming

Geometric programs introduced by [4] are a special type of nonlinear programming problems in which the objective function and/or some of constraints are *posynomials*:

$$\text{minimize } g_0(\boldsymbol{t}) \text{ subject to } g_k(\boldsymbol{t}) \le 1,\ k = 1, \dots, K,\ \boldsymbol{t} \in \mathbb{R}^M_{++} \tag{1}$$

with

$$g_k(\boldsymbol{t}) = \sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} = \sum_{i \in I_k} u_i(\boldsymbol{t}),\ k = 0, \dots, K. \tag{2}$$

We denote Q the total number of *monomials* $u_i(\boldsymbol{t}) = c_i \prod_{j=1}^{M} t_j^{a_{ij}}$ in the formulation of geometric program (1), (2) and $\{I_k, k = 0, \dots, K\}$ is a decomposition of $\{1, \dots, Q\}$ into $K + 1$ disjoint index sets. The exponents a_{ij} are arbitrary real numbers and the coefficients c_i are positive. Notice that simple box inequality constraints can be written as inequalities for monomials.
The special structure of geometric program (1)–(2) allows to derive a numerically tractable *dual problem*:

$$\max_{\delta,\lambda} v(\delta, \lambda) := \prod_{i=1}^{Q} (c_i/\delta_i)^{\delta_i} \prod_{k=1}^{K} \lambda_k^{\lambda_k} \tag{3}$$

subject to

$$\sum_{i \in I_0} \delta_i = 1,\ \delta_i \ge 0,\ i = 1, \dots, Q,$$

$$\sum_{i=1}^{Q} a_{ij}\delta_i = 0,\ j = 1, \dots, M,\ \sum_{i \in I_k} \delta_i = \lambda_k,\ k = 1, \dots, K.$$

The optimal solutions $\boldsymbol{t}^*$ of (1) and δ^*, λ^* of (3) are related as follows:

$$\delta_i^* = \frac{u_i(\boldsymbol{t}^*)}{g_0(\boldsymbol{t}^*)} = \frac{u_i(\boldsymbol{t}^*)}{v(\delta^*, \lambda^*)} \text{ for } i \in I_0$$

$$\delta_i^* = \lambda_k^* u_i(\boldsymbol{t}^*) \text{ for } i \in I_k,\ k = 1, \dots, K.$$

Hence $\frac{\delta_i^*}{\lambda_k^*}$, $i \in I_k$ is the proportional contribution of the i-th monomial to the value of posynomial g_k at the optimal solution $\boldsymbol{t}^*$. Numerical solution of small size geometric programs based on solution of their relatively simple duals exploits these duality relations.

The *degree of difficulty* of a geometric program is defined as $\Delta = Q - M^* - 1$ where M^* denotes the rank of the (Q, M) matrix $\boldsymbol{A} = (a_{ij})$. It refers to the dimensionality of the set of feasible solutions of the dual program. For $\Delta = 0$, i.e. for the *zero degree of difficulty geometric programs*, there is a unique solution of the system $\sum_{i \in I_0} \delta_i = 1$, $\sum_{i=1}^{Q} a_{ij}\delta_i = 0$, $j = 1, \ldots, M$. If this solution is nonnegative, then it is the optimal solution of the dual problem and it is possible to get an explicit representation of the optimal value function of (3) in terms of coefficients c_i. Moreover, its logarithm is a linear function in coefficients c_i.

Geometric programs (GP) can be reformulated as convex programming problems: Using the exponential substitution $z_j = \log t_j \; \forall j$ the posynomials (2) are transformed to

$$h_k(\boldsymbol{z}) = \sum_{i \in I_k} c_i \exp\{\sum_{j=1}^{M} a_{ij} z_j\}, \; k = 0, \ldots, K. \tag{4}$$

An additional log transform of functions h_k is frequently recommended.
The resulting transformed GP is then the *convex* program

$$\text{minimize } h_0(\boldsymbol{z}) \text{ subject to } h_k(\boldsymbol{z}) \le 1, \; k = 1, \ldots, K, \; \boldsymbol{z} \in \mathbb{R}^M. \tag{5}$$

See e.g. [2, 4, 16, 17] for these and related results.
The early applications of geometric programming were connected mainly with mechanical engineering but they include also economic and managerial problems, cf. [16], chemical equilibrium and nonlinear network flow problems. In these areas, more sophisticated applications have been further developed and extended to inventory control, production system optimization, computational finance etc. The presently prevailing field of applications seems to be in digital circuit design.
The recently observed growing interest in GP stems from the fact that various practical problems can be reformulated as geometric programs and there are solution methods which solve even very large-scale GPs efficiently and reliably. With a basic interior-point method which exploits sparsity of the generic geometric program (1)–(2) the reported efficiency is close to that of linear programming solvers. We refer to [3] for an up-to-date survey of various applications and an extensive list of references and to [18] for an interesting reformulation of an entropy optimization problem emanating from computational finance to a dual of a tractable GP.

2 Stochastic geometric programming

In applications, some of coefficients c_i and/or exponents a_{ij} need not be known precisely and their incomplete knowledge may be modeled as random. As in general stochastic programming problems one deals with the distribution problem or focuses on decision problems. The question is which of stochastic programming approaches and under which distributional assumptions do not destroy the favorable structure of the (generalized) geometric programs.

The origins of *stochastic* geometric programming (SGP) are connected with paper [1], where the exponents a_{ij} are deterministic and the coefficients c_j are positive random variables. The main result of the paper are numerically tractable bounds for the optimal value of (1); see also [13, 20] for their further elaboration and application.

Construction of confidence bounds for the optimal value of a geometric program, deriving its moments or probability distribution is a task belonging under *distribution problem* of stochastic geometric programming. It was developed first for zero degree of difficulty geometric programs in connection with lognormal distribution of coefficients c_i and fixed exponents a_{ij}. Then the logarithm of the optimal value function in (3) is an affine linear function in $\log c_i$, hence, for a lognormal distribution of c_i, one gets lognormal distribution of the optimal value. For extensions of these results to other probability distributions and to problems with degree of difficulty $\Delta > 0$ see e.g. [8, 19].

Individual probabilistic constraints have been applied under assumption of deterministic exponents and normally distributed, mostly uncorrelated coefficients c_i; see e.g. [11, 17]. It means that the constraints of (1)

$$\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \le 1, \; k = 1, \ldots, K$$

are replaced by

$$P\{\sum_{i\in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1\} \geq 1-\varepsilon_k,\; k=1,\ldots,K,$$

with prescribed tolerances ε_k. For independent normally distributed coefficients $c_i \sim N(Ec_i, \sigma_i^2)\, \forall i$ these constraints are equivalent to

$$\sum_{i\in I_k} Ec_i \prod_{j=1}^{M} t_j^{a_{ij}} + \Phi^{-1}(1-\varepsilon_k)\sqrt{\sum_{i\in I_k} \sigma_i^2 \prod_{j=1}^{M} t_j^{2a_{ij}}} \leq 1,\; k=1,\ldots,K,$$

where $\Phi^{-1}(1-\varepsilon_k)$ is quantile of the standard normal distribution $N(0,1)$. Each constraint is then split into two constraints that involve posynomials in $t_j \forall j$ and a common additional slack variable.
Of course, the assumption of normally distributed costs c_i is not in agreement with the required positivity of coefficients in (1). For general probability distributions of coefficients c_i [10] suggests to approximate the probabilistic constraints by one-sided Chebyshev inequality. A similar approximation is used also in [14] for optimization of stochastic activity networks with random durations characterized by mean values and standard deviations of the posynomial form.

In various engineering and economic applications of GP random character of exponents can be observed as well. Consider for example production functions of the Cobb-Douglas type used to describe requirements or to formulate the objective function. In the simplest situation, the constraint on production is

$$Ct_1^{a_1} t_2^{a_2} \geq k \tag{6}$$

where t_1, t_2 are inputs. The common assumption that the coefficient C and exponents a_1, a_2 are given constants is not quite realistic. Hence, one gets interested in sensitivity of results on small changes of these "constants"; the classical sensitivity analysis, cf. [15, 16] is the first step. It is not enough, however, when the coefficients *and* exponents of posynomials are random, being e.g. differentiable functions of statistical estimates of true parameter values. In comparison with randomness present only in the coefficients c_i, a substantially higher level of difficulty arises. In general, one can design simulation experiments to get an idea about the probability distribution of the optimal value, to evaluate approximate confidence bounds and moments of the optimal value, etc. However, such experiments are computationally expensive and do not provide sufficient information about the optimal solutions or their logarithms. In the sequel, we shall review some other techniques.

A possibility which applies to SGP with random parameter, say β, only in the objective function and to a discrete distribution of these parameters is to use a *tracking model* related with the goal programming; cf. [6].

For random costs and exponents in the objective function and in constraints of (1), a *penalization or two-stage approach* was suggested in [12]. First of all, using an additional constraint and an additional variable t_0, geometric program (1) can be rewritten to have a nonrandom linear objective function:

$$\min\{t_0 \,:\, t_0^{-1} g_0(\boldsymbol{t}, \beta) \leq 1,\; g_k(\boldsymbol{t}, \beta) \leq 1,\; k=1,\ldots,K,\; t_0 > 0,\; \boldsymbol{t} \in \mathbb{R}^M_{++}\}. \tag{7}$$

The constraints of (7) can be further split to

$$u_i(\boldsymbol{t}, \beta)\theta_{ik}^{-1} \leq 1,\; i=1,\ldots,Q,\; k=0,\ldots,K \tag{8}$$

with $\theta_{ik} > 0$, $\sum_{i\in I_k} \theta_{ik} = 1$ interpreted as the proportional contribution of i-th monomial to the value of k-th posynomial.
The first stage decisions are $t_0 > 0$, $\boldsymbol{t} \in \mathbb{R}^M_{++}$ and $\theta_{ik} > 0$, $i \in I_k \forall k$, and $\sum_{i\in I_k} \theta_{ik} = 1 \forall k$ are the first stage constraints. After observing realizations of random coefficients and exponents, possible violation of constraints (8) can be corrected for an additional cost. Logarithmic penalty function is suggested and the case of multivariate discrete or normal distribution of parameters c_i, a_{ij} is discussed.

For GP of the form (7) one may consider a *robust reformulation*

$$\min\{t_0 \,:\, t_0^{-1} g_0(\boldsymbol{t}, \boldsymbol{u}) \leq 1,\; g_k(\boldsymbol{t}, \boldsymbol{u}) \leq 1,\; k=1,\ldots,K,\; t_0 > 0,\; \boldsymbol{t} \in \mathcal{T},\; \boldsymbol{u} \in \mathcal{U}\}$$

where $\mathcal{U}$ denotes a prespecified uncertainty set. Various possibilities how to approach such semiinfinite problems are discussed e.g. in [9].

In [6, 7] a technique for construction of *confidence bounds* for optimal value and optimal solution of SGP has been proposed. It is based on sensitivity results for deterministic geometric programming [15] and on stochastic sensitivity analysis [5]. The motivation comes from metal cutting problems where the tool life affects substantially the total machining costs. Due to nonhomogeneity of the machined and cutting material variability of the tool life occurs even at fixed machining conditions. It can be influenced by a careful choice of cutting conditions in accordance with the postulated technical relation: The tool life is a monomial in cutting speed, feed and depth whose parameters can be obtained as statistical estimates of the true values. Derivatives of the minimal total cost with respect to the parameters, regression analysis and Delta theorem lead to an approximate confidence interval for the minimal machining costs, the tool life, etc.
This technique can be evidently applied to decision problems involving estimated production, demand or utility functions of the posynomial form such as the Cobb-Douglas production function in (6); see [16] for instances of deterministic versions of such problems. Among others, the lower and upper bounds on the system's cost obtained in this way are an important information for the purpose of economic decision making.

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