Risk Measures and Multistage Stochastic Programming

Václav Kozmík

Faculty of Mathematics and Physics
Charles University in Prague

21. 4. 2011

Introduction

Definition

Let $(\Omega, \mathcal{F})$ be a sample space on which uncertain losses $Z(\omega)$ are defined. For some space $\mathcal{Z}$ of functions Z we understand risk measure as a function $\rho(Z)$ which maps $\mathcal{Z}$ into extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$.

- We usually have $\mathcal{Z} = \mathcal{L}_p$ with $p \in [1, \infty)$.
- We assume that ρ is proper, i.e. $\rho(Z) \geq -\infty \; \forall Z \in \mathcal{Z}$ and the domain $\mathrm{dom}(\rho) = \{Z \in \mathcal{Z} : \rho(Z) < \infty\} \neq \emptyset$
- $Z, Z^{'} \in \mathcal{Z}$, $Z \succeq Z^{'}$ if $Z(\omega) \geq Z^{'}(\omega)$ for a.e. $\omega \in \Omega$.
- The smaller Z is better, representing for instance costs.

Risk measures - examples

Let $H_Z(z) = \mathsf{P}[Z < z]$ and denote the left side quantile $H_Z^{-1}(\alpha) = \inf\{t : H_Z(t) \geq \alpha\}$.

- variance $\mathrm{var}(Z) = \mathsf{E}[Z - \mathsf{E}Z]^2$
- semideviations $\sigma_p^+(Z) = \left(\mathsf{E}[Z - \mathsf{E}Z]_+^p\right)^{1/p}$
- Value at Risk $\mathrm{VaR}_\alpha(Z) = H_Z^{-1}(1-\alpha)$
- Condition Value at Risk $\mathrm{CVaR}_\alpha(Z) = \inf_{t\in\mathbb{R}}\left\{t + \alpha^{-1}\mathsf{E}[Z-t]_+\right\}$
- weighted mean deviation from a quantile $q_\alpha(Z) = \mathsf{E}\left[\max\left\{(1-\alpha)(H_Z^{-1}(\alpha) - Z), \alpha(Z - H_Z^{-1}(\alpha)\right\}\right]$
- etc.

Coherent Risk Measures

Definition

Risk measure ρ is said to be coherent if it satisfies:

1. *Convexity:* $\forall Z, Z^{'} \in \mathcal{Z}$ *and* $\forall t \in [0,1]$

$$\rho(tZ + (1-t)Z^{'}) \leq t\rho(Z) + (1-t)\rho(Z^{'}).$$

2. *Monotonicity: if* $Z, Z^{'} \in \mathcal{Z}$ *and* $Z \succeq Z^{'}$ *then* $\rho(Z) \geq \rho(Z^{'})$.
3. *Translation equivariance:* $\forall a \in \mathbb{R},\ Z \in \mathcal{Z}$: $\rho(Z + a) = \rho(Z) + a$
4. *Positive homogeneity:* $\forall t > 0,\ Z \in \mathcal{Z}$: $\rho(tZ) = t\rho(Z)$

- CVaR is an example of coherent risk measure.

Conjugate duality

- With each space $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$ is associated its dual $\mathcal{Z}^* = \mathcal{L}_q(\Omega, \mathcal{F}, \mathrm{P})$ where $q \in [1, \infty)$ such that $1/p + 1/q = 1$.
- Scalar product for $Z \in \mathcal{Z}$ and $\zeta \in \mathcal{Z}^*$ is given by

$$\langle \zeta, Z \rangle = \int_\Omega \zeta(\omega) Z(\omega) \mathrm{dP}(\omega)$$

- Conjugate function $\rho^*(\zeta)$ is defined as:

$$\rho^*(\zeta) = \sup_{Z \in \mathcal{Z}} \{\langle \zeta, Z \rangle - \rho(Z)\}$$

  - always convex and lsc
- Biconjugate function $\rho^{**}(Z)$, which is conjugate of $\rho^*(Z)$:

$$\rho^{**}(Z) = \sup_{\zeta \in \mathcal{Z}^*} \{\langle \zeta, Z \rangle - \rho^*(\zeta)\}$$

Conjugate duality

Theorem (Fenchel-Moreau)

Let $\mathcal{Z}$ be a Banach space and $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ be a proper extended real valued convex function. Then

$$\rho^{**} = \operatorname{lsc} \rho.$$

- If ρ is convex, proper and lower semicontinuous then ρ^* is proper and $\rho^{**} = \rho$.
- We can use following equivalent form for convex risk measure:

$$\rho(Z) = \sup_{\zeta \in \mathcal{U}} \{\langle \zeta, Z \rangle - \rho^*(\zeta)\}$$

where $\mathcal{U} = \operatorname{dom}(\rho^*)$.

Basic duality theorem

Theorem

Let $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ *be convex, proper and lsc. Then for* $\mathcal{U} = \mathrm{dom}(\rho^*)$ *representation*

$$\rho(Z) = \sup_{\zeta \in \mathcal{U}} \{\langle \zeta, Z \rangle - \rho^*(\zeta)\}$$

holds. Moreover

- ρ *is monotone iff* $\forall \zeta \in \mathcal{U} : \zeta(\omega) \geq 0$ *a.s.,*
- ρ *is translation equivariant iff* $\forall \zeta \in \mathcal{U} : \int_\Omega \zeta \mathrm{dP} = 1$,
- ρ *is positive homogeneous iff* ρ *is the support function of the set* $\mathcal{U}$, *i.e.*

$$\rho(Z) = \sup_{\zeta \in \mathcal{U}} \langle \zeta, Z \rangle.$$

Basic duality theorem - proof

- Representation

$$\rho(Z) = \sup_{\zeta \in \mathcal{U}} \{\langle \zeta, Z \rangle - \rho^*(\zeta)\}$$

  follows from Fenchel-Moreau theorem.
- Suppose ρ is monotone.
  - If $\zeta \in \mathcal{Z}^*$ is not nonnegative, then $\exists \Delta \in \mathcal{F} : \mathsf{P}[\Delta] > 0, \zeta(\omega) < 0 \ \forall \omega \in \Delta$.
  - Define $\hat{Z} = \mathsf{I}_\Delta$, then $\langle \zeta, \hat{Z} \rangle < 0$.
  - For any $Z \in \text{dom}(\rho)$ define $Z_t = Z - t\hat{Z}$.
  - Then $\rho^*(\zeta) \geq \sup_{t \in \mathbb{R}_+} \{\langle \zeta, Z_t \rangle - \rho(Z_t)\} \geq \sup_{t \in \mathbb{R}_+} \left\{\langle \zeta, Z \rangle - t\langle \zeta, \hat{Z} \rangle - \rho(Z_t)\right\} = \infty$
- Suppose every $\zeta \in \mathcal{U}$ is nonnegative.
  - $\forall \zeta \in \mathcal{U}$ and $Z \succeq Z^{'}$ we have $\langle \zeta, Z \rangle \geq \langle \zeta, Z^{'} \rangle$
  - That means if $Z \succeq Z^{'}$ then $\rho(Z) \geq \rho(Z^{'})$

Basic duality theorem - proof

- Suppose ρ is translation equivariant
  - $\forall Z \in \mathrm{dom}(\rho)$: $\rho^*(\zeta) \geq \sup_{a\in\mathbb{R}} \{\langle \zeta, Z + a\rangle - \rho(Z + a)\} \geq \sup_{a\in\mathbb{R}} \left\{a \int_\Omega \zeta \mathrm{dP} - a + \langle \zeta, Z\rangle - \rho(Z)\right\}$
  - If $\int_\Omega \zeta \mathrm{dP} \neq 1$ then $\rho^*(\zeta) = \infty$
- Conversely $\int_\Omega \zeta \mathrm{dP} = 1$
  - $\rho(Z + a) = \sup_{\zeta\in\mathcal{U}} \{\langle \zeta, Z + a\rangle - \rho^*(\zeta)\} = \sup_{\zeta\in\mathcal{U}} \{\langle \zeta, Z\rangle + a - \rho^*(\zeta)\} = \rho(Z) + a$
- Suppose ρ is positive homogeneous
  - $\rho^*(\zeta) = c > 0$ then $\rho^*(\zeta) \geq \sup_{t\in\mathbb{R}_+} \{\langle \zeta, tZ\rangle - \rho(tZ)\} = \sup_{t\in\mathbb{R}_+} \{t\langle \zeta, Z\rangle - t\rho(Z)\} = \infty$
  - $\rho^*(\zeta)$ is indicator function of some convex set
  - we conclude $\rho^*(\zeta) = 0$ for $\zeta \in \mathcal{U}$ (basic representation)
- Conversely
  - $\rho(tZ) = \sup_{\zeta\in\mathcal{U}} \langle \zeta, tZ\rangle = \sup_{\zeta\in\mathcal{U}} t\langle \zeta, Z\rangle = t\rho(Z)$

Basic duality theorem - corollary

Let ρ be a coherent risk measure. Then

$$\rho(Z) = \sup_{\zeta \in \mathcal{U}} \langle \zeta, Z \rangle,$$

where $\mathcal{U}$ is a set of probability density functions.
Consequently we can write

$$\rho(Z) = \sup_{\zeta \in \mathcal{U}} \mathsf{E}_{\zeta} \left[Z \right].$$

Basic duality theorem - examples

- Conditional Value at Risk ($\mathcal{Z} = \mathcal{L}_1, \mathcal{Z}^* = \mathcal{L}_\infty$)

$$\mathrm{CVaR}_\alpha(Z) = \sup_{\zeta \in \mathcal{U}} \langle \zeta, Z \rangle,$$

$$\mathcal{U} = \left\{ \zeta \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P}) : \zeta(\omega) \in [0, \alpha^{-1}] \ a.s., \mathrm{E}[\zeta] = 1 \right\}$$

- Mean-Variance Risk Measure ($\mathcal{Z}^* = \mathcal{Z} = \mathcal{L}_2$)

$$\rho(Z) = \mathrm{E}[Z] + k \mathrm{var}[Z]$$

$$\rho(Z) = \sup \left\{ \langle \zeta, Z \rangle - \frac{1}{4k} \mathrm{var}[\zeta] : \zeta \in \mathcal{Z}, \mathrm{E}[\zeta] = 1 \right\}$$

Extensions to multiperiod case

- Conditional risk mappings
  - details in Shapiro, A., Dentcheva, D., Ruszczynski A. (2009)
  - good interpretation on the scenario tree
- Multiperiod coherent risk measures
  - details in Shapiro, A., Dentcheva, D., Ruszczynski A. (2009)
  - general framework for multiperiod risk-averse optimization
- Multiperiod polyhedral risk measures
  - special class with nice properties and good tractability
  - details in Eichhorn, A. and Romisch W. (2005), Guigues, V. and Romisch W. (2010)

Conditional risk mappings

Definition

Let Ω be the sample space equipped with sigma algebras $\mathcal{F}_t, \mathcal{F}_{t+1}$ and a probability measure P *on $(\Omega, \mathcal{F}_{t+1})$. Denote spaces $\mathcal{Z}_t = \mathcal{L}_p(\Omega, \mathcal{F}_t, \mathrm{P})$ and $\mathcal{Z}_{t+1} = \mathcal{L}_p(\Omega, \mathcal{F}_{t+1}, \mathrm{P})$. Mapping $\rho : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ is conditional risk mapping if it satisfies*

1. *Convexity:* $\forall Z, Z' \in \mathcal{Z}_{t+1}$ *and* $\forall t \in [0,1]$
$$\rho(tZ + (1-t)Z') \preceq t\rho(Z) + (1-t)\rho(Z').$$
2. *Monotonicity: if* $Z, Z' \in \mathcal{Z}_{t+1}$ *and* $Z \succeq Z'$ *then* $\rho(Z) \succeq \rho(Z')$.
3. *Translation equivariance:* $\forall Y \in \mathcal{Z}_t,\ Z \in \mathcal{Z}_{t+1}$*:* $\rho(Z + Y) = \rho(Z) + Y$
4. *Positive homogeneity:* $\forall t > 0,\ Z \in \mathcal{Z}_{t+1}$*:* $\rho(tZ) = t\rho(Z)$

Conditional risk mappings - examples

- Conditional expectation $\mathrm{E}\left[\cdot|\mathcal{F}_t\right]$
  - $Z \in \mathcal{L}_p(\Omega, \mathcal{F}_{t+1}, \mathrm{P})$, $\mathcal{F}_t$-measurability of $\mathrm{E}\left[Z|\mathcal{F}_t\right]$ is clear

$$\int_\Omega \left|\mathrm{E}\left[Z|\mathcal{F}_t\right]\right|^p \mathrm{dP} \leq \int_\Omega \mathrm{E}\left[|Z|^p|\mathcal{F}_t\right] \mathrm{dP} = \mathrm{E}\left[|Z|^p\right] < \infty$$

- Conditional CVaR

$$\left[\mathrm{CVaR}_\alpha(Z|\mathcal{F}_t)\right](\omega) = \inf_{Y \in \mathcal{Z}_t} \left\{ Y(\omega) + \alpha^{-1} \mathrm{E}\left[(Z - Y)_+ | \mathcal{F}_t\right](\omega) \right\}$$

- Conditional absolute semideviation:

$$\rho_{d|\mathcal{F}_t}(Z) = \mathrm{E}\left[Z|\mathcal{F}_t\right] + \mathrm{E}\left[\left(Z - \mathrm{E}\left[Z|\mathcal{F}_t\right]\right)_+ |\mathcal{F}_t\right]$$

Conditional risk mappings - usage

- Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_T$ sequence of sigma algebras with $\mathcal{F}_1 = \{\emptyset, \Omega\}, \mathcal{F}_T = \mathcal{F}$. Let $\mathcal{Z}_t = \mathcal{L}_p(\Omega, \mathcal{F}_t, \mathrm{P})$ and $\rho_{t+1|\mathcal{F}_t} : \mathcal{Z}_{t+1} \to \mathcal{Z}_t,\ t = 1, .., T-1.$
- Consider following multistage program

$$
\begin{aligned}
\min_{x_1 \in \mathcal{X}_1} f_1(x_1) &+ \rho_{2|\mathcal{F}_1}\left(\inf_{x_2 \in \mathcal{X}_2(x_1,\omega)} f_2(x_2,\omega) + \cdots \right. \\
&+ \rho_{T-1|\mathcal{F}_{T-2}}\left(\inf_{x_{T-1} \in \mathcal{X}_{T-1}(x_{T-2},\omega)} f_{T-1}(x_{T-1},\omega) \right. \\
&\left.\left.\left. + \rho_{T|\mathcal{F}_{T-1}}\left(\inf_{x_T \in \mathcal{X}_T(x_{T-1},\omega)} f_T(x_T,\omega) \right) \right) \right)
\end{aligned}
$$

Conditional risk mappings - usage

- Denote $Z_t = \inf_{x_t \in \mathcal{X}_t(x_{t-1},\omega)} f_t(x_t, \omega)$
- By translation equivariance we have:

$$\rho_{T-1|\mathcal{F}_{T-2}}\left(Z_{T-1} + \rho_{T|\mathcal{F}_{T-1}}(Z_T)\right) = \rho_{T-1|\mathcal{F}_{T-2}} \circ \rho_{T|\mathcal{F}_{T-1}}(Z_{T-1} + Z_T)$$

- Applying the same way we get coherent risk measure

$$\overline{\rho} = \rho_{2|\mathcal{F}_1} \circ \cdots \circ \rho_{T|\mathcal{F}_{T-1}}$$

- Stochastic program using the composite measure

$$\min_{x_1,\mathbf{x}_2,\ldots,\mathbf{x}_T} \overline{\rho}\left(f_1(x_1) + f_2(\mathbf{x}_2,\omega) + \cdots + f_T(\mathbf{x}_T,\omega)\right)$$
$$\text{s. t. } x_1 \in \mathcal{X}_1,\ \mathbf{x}_t \in \mathcal{X}_t(\mathbf{x}_{t-1},\omega),\ t = 2,\ldots,T$$

Multiperiod coherent risk measures

Denote $\mathcal{Z} = \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_T$ and its dual $\mathcal{Z}^* = \mathcal{Z}_1^* \times \cdots \times \mathcal{Z}_T^*$.

Definition

We say that $\rho : \mathcal{Z} \to \mathbb{R}$ is a multiperiod coherent risk measure if it satisfies

1. *Convexity:* $\forall Z, Z' \in \mathcal{Z}$ *and* $\forall t \in [0,1]$
   $\rho(tZ + (1-t)Z') \leq t\rho(Z) + (1-t)\rho(Z')$.
2. *Monotonicity: if* $Z, Z' \in \mathcal{Z}$ *and* $Z \succeq Z'$ *(componentwise) then* $\rho(Z) \geq \rho(Z')$.
3. *Translation equivariance:*
   $\forall Z = (Z_1, \ldots, Z_T) \in \mathcal{Z},\ Y_t \in \mathcal{Z}_t,\ a \in \mathbb{R}$:
   $\rho(Z_1, ..Z_t, Z_{t+1} + Y_t, .., Z_T) = \rho(Z_1, .., Z_t + Y_t, Z_{t+1}, ..Z_T)$
   $\rho(Z_1 + a, .., Z_T) = \rho(Z_1, .., Z_T) + a$
4. *Positive homogeneity:* $\forall t > 0,\ Z \in \mathcal{Z}$: $\rho(tZ) = t\rho(Z)$

Multiperiod coherent risk measures

Theorem

Let $\rho : \mathcal{Z} \to \mathbb{R}$ be a multiperiod coherent risk measure. Then there exists a coherent risk measure $\overline{\rho} : \mathcal{Z}_T \to \mathbb{R}$ such that

$$\rho(Z_1, \dots, Z_T) = \overline{\rho}(Z_1 + \dots + Z_T).$$

Moreover there exists nonempty, bounded set $\mathcal{U}_T \subset \mathcal{Z}_T^*$ of probability density functions such that dual representation

$$\rho(Z) = \sup_{\zeta \in \mathcal{U}} \langle \zeta, Z \rangle$$

holds with corresponding set $\mathcal{U}$ of the form

$$\mathcal{U} = \{(\zeta_1, .., \zeta_T) : \zeta_T \in \mathcal{U}_T,\ \zeta_t = \mathsf{E}\left[\zeta_T | \mathcal{F}_t\right]\}$$

Multiperiod coherent risk measures - examples

- Linear combination of CVaR:

$$\rho(Z_1, \ldots, Z_T) = \sum_{t=1}^{T} \lambda_t \mathrm{CVaR}_\alpha(Z_t)$$

  with weights $\sum_{t=1}^{T} \lambda_t = 1$

- Maximal risk of all stages using CVaR:

$$\rho(Z_1, \ldots, Z_T) = \mathrm{CVaR}_\alpha \left(\min_{t=1,\ldots,T} Z_t \right)$$

Polyhedral risk measures

Definition

Risk measure $\rho : \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}) \to \overline{\mathbb{R}}$ is called polyhedral if there exist $k_1, k_2 \in \mathbb{N}$, $c_1, w_1 \in \mathbb{R}^{k_1}$ $c_2, w_2 \in \mathbb{R}^{k_2}$, a nonempty polyhedral set $M_1 \subset \mathbb{R}^{k_1}$ and polyhedral cone $M_2 \subset \mathbb{R}^{k_2}$ such that:

$$\rho(Z) = \inf \Big\{ \langle c_1, Y_1 \rangle + \mathrm{E}\left[\langle c_2, Y_2 \rangle\right] : \\ Y_1 \in M_1, \\ Y_2 \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}),\, Y_2 \in M_2 \text{ a.s.}, \\ \langle w_1, Y_1 \rangle + \langle w_2, Y_2 \rangle = Z \text{ a.s.} \Big\}$$

for every $Z \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$.

Polyhedral risk measures - examples

- Conditional Value at Risk

$$\mathrm{CVaR}_\alpha(Z) = \inf \left\{ Y_1 + \frac{1}{\alpha}\mathrm{E}\left[Y_2^{(1)}\right] : \begin{array}{l} Y_1 \in \mathbb{R}, Y_2 \in \mathcal{L}_1(\Omega, \mathcal{F}, \mathrm{P}) \\ Y_2 \in \mathbb{R}_+ \times \mathbb{R}_+ \ a.s., \\ Y_2^{(1)} - Y_2^{(2)} = Z - Y_1 \ a.s. \end{array} \right\}$$

- Expected loss $\mathrm{E}\left[Z - \gamma\right]_+$ for some fixed γ.
- Dispersion measures

$$d_\alpha(Z) = \mathrm{E}\left[\alpha(Z - q_\alpha)_- + (1-\alpha)(Z - q_\alpha)_+\right]$$

Multiperiod polyhedral risk measures

Definition

Multiperiod risk measure ρ *on* $\times_{t=1}^{T}\mathcal{L}_p(\Omega,\mathcal{F},\mathrm{P})$ *is called multiperiod polyhedral if there exist* $k_t \in \mathbb{N},\ c_t \in \mathbb{R}^{k_t}, t = 1,\dots,T$, $w_{t\tau} \in \mathbb{R}^{k_{t-\tau}}, t = 2,\dots,T, \tau = 0,\dots,t-1$, *a polyhedral set* $M_1 \subset \mathbb{R}^{k_1}$ *and polyhedral cones* $M_t \subset \mathbb{R}^{k_t}, t = 2,\dots,T$ *such that:*

$$\rho(Z) = \inf\Bigg\{ \mathrm{E}\left[\sum_{t=1}^{T}\langle c_t, Y_t\rangle\right] : \\ Y_t \in M_t \text{ a.s.}, Y_t \in \mathcal{L}_p(\Omega,\mathcal{F},\mathrm{P}), t = 1,\dots,T, \\ \sum_{\tau=0}^{t-1}\langle w_{t\tau}, Y_{t-\tau}\rangle = Z_t \text{ a.s.}, t = 1,\dots,T \Bigg\}$$

Multiperiod polyhedral risk measures - example

- Consider following risk averse problem:

$$\inf \rho \left(f_1(x_1, \xi_1), \sum_{\tau=1}^{2} f(x_\tau, \xi_\tau), \dots, \sum_{\tau=1}^{T} f(x_\tau, \xi_\tau) \right)$$

$$x_t \in \mathcal{X}_t \left(x_{t-1}, \xi_t\right), t = 1, \dots, T$$

- Let ρ be defined as:

$$\rho(Z_1, \dots, Z_T) = \lambda_1 \mathsf{E}\left[Z_T\right] + \sum_{t=2}^{T} \lambda_t \mathrm{CVaR}_\alpha(Z_t)$$

with $\sum_{t=1}^{T} \lambda_t = 1$

- Using the fact that CVaR is coherent we get:

$$\rho(\cdot) = Z_1 + \lambda_1 \mathsf{E}\left[Z_T - Z_1\right] + \sum_{t=2}^{T} \lambda_t \mathrm{CVaR}_\alpha(Z_t - Z_1)$$

Multiperiod polyhedral risk measures - example

- And after some evaluation we get dynamic programming equations:

$$\inf_{x_1, w_2, \dots w_T} f(x_1, \xi_1) + \sum_{t=2}^{T} \lambda_t w_t + \mathcal{Q}_2(x_1, \xi_{[1]}, \overline{Z}_1, w_2, \dots, w_T)$$

$$x_1 \in \mathcal{X}_1(x_0, \xi_1), w_t \in \mathbb{R}, t = 2, \dots, T$$

where $\overline{Z}_1 = 0$ and for $t = 2, \dots, T$:

$$\mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}, \overline{Z}_{t-1}, w_t, \dots, w_T) =$$

$$\mathsf{E}_{\xi_t | \xi_{[t-1]}} \Bigg[\inf_{x_t, \overline{Z}_t} \delta_{tT} \lambda_1 \overline{Z}_t + \frac{\lambda_t}{\alpha} (w_t - \overline{Z}_t)_+ + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}, \overline{Z}_t, w_{t+1}, \dots, w_T)$$

$$\overline{Z}_t = \overline{Z}_{t-1} + f_t(x_t, \xi_t), x_t \in \mathcal{X}_t(x_{t-1}, \xi_t) \Bigg]$$

References

- Eichhorn, A. and Romisch W. (2005): *Polyhedral Risk Measures in Stochastic Programming*, SIAM Journal on Optimization, 16 (1), pp. 69-95
- Guigues, V. and Romisch W. (2010): *Sampling-based decomposition methods for risk-averse multistage stochastic programs*, Optimization Online
- Shapiro, A., Dentcheva, D., Ruszczynski A. (2009): *Lectures on Stochastic Programming: Modeling and Theory*, SIAM-Society for Industrial and Applied Mathematics, ISBN 978-0898716870

Conclusion

Thank you for your attention!

Václav Kozmík
vkozmik@gmail.com