

Diferencovatelnost reálných funkcí

Petr Lachout

KPMS - 31.března 2016



○○○○○

Diferencovatelnost reálných funkcí

Definition 1

Let $D \subset \mathbb{R}$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$ and $x \in \text{int}(D)$. We say,
 f is differentiable at x (cz. diferencovatelná v bodě x) if there is
 $f'(x) \in \mathbb{R}$ such that for all $y \in D$ we have

$$f(y) = f(x) + f'(x)(y - x) + |y - x| R_1(y - x; f, x), \quad (1)$$

where $\lim_{h \rightarrow 0} R_1(h; f, x) = 0$.

Equivalently, f is differentiable at x iff $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} = f'(x) \in \mathbb{R}$.
If $S \subset \text{int}(D)$, then we say f is differentiable at S (cz. diferencovatelná v množině S), if it is differentiable at each point $x \in S$.



○○○○○○

Jeden argument

Lemma 2

If $D \subset \mathbb{R}$, $D \neq \emptyset$ and $f : D \rightarrow \mathbb{R}$ is differentiable at $x \in \text{int}(D)$ then f is continuous at x .

Lemma 3

Let $a, b \in \mathbb{R}$, $a < b$, $f : [a, b] \rightarrow \mathbb{R}$ be differentiable at (a, b) , right-continuous at a and left-continuous at b . Then,

$$\int_a^b f'(s) \, ds = f(b) - f(a). \quad (2)$$



○○○○○

Více argumentů - 0

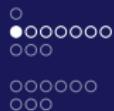
Definition 4

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$, $x \in \text{int}(D)$ and $h \in \mathbb{R}^n$. We say, f is differentiable at x in direction h (cz. diferencovatelná v bodě x ve směru h) if there is $f'(x; h) \in \mathbb{R}$ such that for all $t \in \mathbb{R}$, $x + th \in D$ we have

$$f(x + th) = f(x) + f'(x; h)t + |t|R_1(t; f, x, h), \quad (3)$$

where $\lim_{s \rightarrow 0} R_1(s; f, x, h) = 0$.

Equivalently, f is differentiable at x in direction h iff
 $\lim_{t \rightarrow 0} \frac{f(x+th)-f(x)}{t} = f'(x; h) \in \mathbb{R}$.



○○○○○

Více argumentů - 1

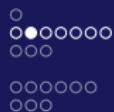
Definition 5

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$, $x \in \text{int}(D)$. For $i \in \{1, 2, \dots, n\}$, we say f possesses a partial derivative at x w.r.t. x_i (cz. parciální derivace v bodě x vzhledem k x_i) if f is differentiable at x in direction $e_{i:n}$ and we set

$$\frac{\partial f}{\partial x_i}(x) = f'(x; e_{i:n}).$$

If f possesses a partial derivative at x w.r.t. x_i for all $i \in \{1, 2, \dots, n\}$ we say f possesses a gradient at x (cz. gradient v bodě x) and we denote

$$\nabla_x f(x) = \left(\frac{\partial f}{\partial x_i}(x) \right)_{i=1}^n.$$



○○○○○

Více argumentů - 2

Definition 6

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$ and $x \in \text{int}(D)$. We say,

f is differentiable at x (or, possesses total differential at x , Fréchet differentiable at x) (cz. diferencovatelná v bodě x) if f possesses a gradient $\nabla_x f(x) \in \mathbb{R}^n$ and for all $y \in D$ we have

$$f(y) = f(x) + \langle \nabla_x f(x), y - x \rangle + \|y - x\| R_1(y - x; f, x), \quad (4)$$

where $\lim_{h \rightarrow 0} R_1(h; f, x) = 0$.

If $S \subset \text{int}(D)$, then we say f is differentiable at S (cz. diferencovatelná v množině S), if it is differentiable at each point $x \in S$.



○○○○○

Více argumentů - 3

Definition 7

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$ and $x \in \text{int}(D)$.

We say, f is continuously differentiable at x (cz. spojité diferencovatelná v bodě x), if there is $\delta > 0$ such that $\mathcal{U}(x, \delta) \subset D$, f is differentiable at $\mathcal{U}(x, \delta)$ and gradient $\nabla_x f$ is continuous at x .

We say, f is continuously differentiable at a neighborhood of x (cz. spojité diferencovatelná v okolí bodu x), if there is $\delta > 0$ such that $\mathcal{U}(x, \delta) \subset D$, f is differentiable at $\mathcal{U}(x, \delta)$ and gradient $\nabla_x f$ is continuous at $\mathcal{U}(x, \delta)$.



○○○○○

Více argumentů - 4

Gradient is necessary for expansion (4).

Lemma 8

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$ and $x \in \text{int}(D)$. Let f fulfills an expansion for all $y \in D$

$$f(y) = f(x) + \langle \xi, y - x \rangle + \|y - x\| R_1(y - x; f, x), \quad (5)$$

where $\xi \in \mathbb{R}^n$ and $\lim_{h \rightarrow 0} R_1(h; f, x) = 0$.

Then f is differentiable at x , $\xi = \nabla_x f(x)$ and $f'(x; h) = \langle \nabla_x f(x), h \rangle$ for all directions $h \in \mathbb{R}^n$.



○○○○○

Více argumentů - 4hint

Důkaz: Using (5) for a direction $h \in \mathbb{R}^n$ and $t \in \mathbb{R}$ small enough, we have

$$f(x + th) = f(x) + \langle \xi, th \rangle + \|th\| R_1(th; f, x),$$

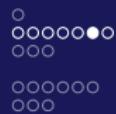
where $\lim_{h \rightarrow 0} R_1(h; f, x) = 0$.

Consider derivative ratio and let $t \rightarrow 0$:

$$\lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t} = \langle \xi, h \rangle + \|h\| \lim_{t \rightarrow 0} \frac{|t|}{t} R_1(th; f, x) = \langle \xi, h \rangle.$$

Setting $h = e_{i:n}$, we receive $\xi_i = \frac{\partial f}{\partial x_i}(x)$.

We have verified ξ is the gradient of f at x , f is differentiable at x and directional derivatives possess announced form.



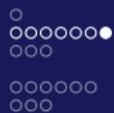
Více argumentů - 5

Lemma 9

If $D \subset \mathbb{R}^n$, $D \neq \emptyset$ and $f : D \rightarrow \mathbb{R}$ is differentiable at $x \in \text{int}(D)$ then f is continuous at x .

Důkaz: Continuity of f at x follows immediately (4).

Q.E.D.



○○○○○○

○○○○○○

Více argumentů - 6

Lemma 10

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$ and $f : D \rightarrow \mathbb{R}$. Consider $x \in D$ and $h \in \mathbb{R}^n$ such that $x + th \in D$ for all $0 \leq t \leq 1$. Define function

$$\varphi : [0, 1] \rightarrow \mathbb{R} : t \in [0, 1] \rightarrow f(x + th).$$

- (i) If $0 < t < 1$, $x + th \in \text{int}(D)$ and f is differentiable at $x + th$ then φ is differentiable at t and $\varphi'(t) = \langle \nabla_x f(x + th), h \rangle$.
- (ii) If $x + th \in \text{int}(D)$ and f is differentiable at $x + th$ for all $0 < t < 1$, φ is continuous at 0 from right and φ is continuous at 1 from left then

$$f(x + h) - f(x) = \varphi(1) - \varphi(0) = \int_0^1 \langle \nabla_x f(x + th), h \rangle dt.$$



○○○○○○○

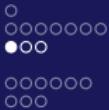
Vektorové funkce - 0

Start with a curve.

Definition 11

Let $D \subset \mathbb{R}$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}^m$ and $t \in \text{int}(D)$. Express the function as a vector of functions $f = (f_1, f_2, \dots, f_m)^\top$. We say,

- ▶ f is differentiable at x if f_j is differentiable at x for each $j \in \{1, 2, \dots, m\}$. We denote the derivative by $f'(t) = (f'_1(t), f'_2(t), \dots, f'_m(t))^\top$.
- ▶ If $S \subset \text{int}(D)$, f is differentiable at S if f_j is differentiable at S for each $j \in \{1, 2, \dots, m\}$.



Vektorové funkce - 1

And now a general case. We start with a notion of multidimensional scalar product.

Definition 12

Let $n, m \in \mathbb{N}$, $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times m}$. We define denotation

$$\langle A, x \rangle = (\langle A_{\cdot,1}, x \rangle, \langle A_{\cdot,2}, x \rangle, \dots, \langle A_{\cdot,m}, x \rangle)^{\top}.$$

Using matrix notation, we can write $\langle A, x \rangle = A^{\top}x$.



Vektorové funkce - 2

Definition 13

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $n \geq 2$, $f : D \rightarrow \mathbb{R}^m$ and $x \in \text{int}(D)$. Express the function as a vector of functions $f = (f_1, f_2, \dots, f_m)^\top$. We say,

- ▶ f possesses a gradient at x if f_j possesses a gradient at x for each $j \in \{1, 2, \dots, m\}$. We denote
$$\nabla_x f(x) = (\nabla_x f_1(x), \nabla_x f_2(x), \dots, \nabla_x f_m(x)).$$
- ▶ f is differentiable at x if f_j is differentiable at x for each $j \in \{1, 2, \dots, m\}$.
- ▶ If $S \subset \text{int}(D)$, f is differentiable at S if f_j is differentiable at S for each $j \in \{1, 2, \dots, m\}$.



Vektorové funkce - 3

Lemma 14

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}^m$ and $x \in \text{int}(D)$. Then, f is differentiable at x if and only if f possesses a gradient $\nabla_x f(x) \in \mathbb{R}^{n \times m}$ and for all $y \in D$ we have

$$f(y) = f(x) + \langle \nabla_x f(x), y - x \rangle + \|y - x\| R_1(y - x; f, x), \quad (6)$$

where $R_1(\cdot; f, x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\lim_{h \rightarrow 0} R_1(h; f, x) = 0$.

The expression is more simple for $n = 1$. Let $D \subset \mathbb{R}$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}^m$ and $t \in \text{int}(D)$. Then, f is differentiable at t if and only if f possesses a derivative $f'(t) \in \mathbb{R}^m$ and for all $s \in D$ we have

$$f(s) = f(t) + (s - t)f'(t) + |s - t| R_1(s - t; f, t), \quad (7)$$

where $R_1(\cdot; f, x) : \mathbb{R} \rightarrow \mathbb{R}^m$ and $\lim_{h \rightarrow 0} R_1(h; f, x) = 0$.



○○○○○○○

Řetízkové pravidlo - 0

Differentiability directly implies chain rule (cz. řetízkové pravidlo).

Lemma 15

Let $I \subset \mathbb{R}$, $\text{int}(I) \neq \emptyset$, $D \subset \mathbb{R}^n$, $\text{int}(D) \neq \emptyset$, $g : I \rightarrow D$, $f : D \rightarrow \mathbb{R}$ and $t \in \text{int}(I)$ such that $g(t) \in \text{int}(D)$. If f is differentiable at $g(t)$ and g is differentiable at t , then $f \circ g$ is differentiable at t and

$$(f \circ g)'(t) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(g(t)) g'_i(t) = \langle \nabla_x f(g(t)), g'(t) \rangle. \quad (8)$$



○○○○○

Druhá derivace - 0

Also, notion of second derivative must be explained.

Definition 16

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$ and $x \in \text{int}(D)$. We say, f possesses second partial derivatives at x (cz. má druhé parciální derivace v x), if f possesses a gradient on a neighborhood of x and all partial derivatives of gradient at x exists; i.e. $\frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right) (x)$ exists for all indexes $i, j \in \{1, 2, \dots, n\}$.

Then, we denote $\frac{\partial^2 f}{\partial x_i \partial x_j} (x) = \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right) (x)$ for all $i, j \in \{1, 2, \dots, n\}$.

Matrix of second partial derivatives is denoted

$\nabla_{x,x}^2 f (x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} (x) \right)_{i=1, j=1}^{n,n}$ and called Hessian matrix.



Druhá derivace - 1

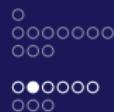
Definition 17

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$ and $x \in \text{int}(D)$. We say, f is twice differentiable at x (cz. dvakrát diferencovatelná v x), if there is a gradient $\nabla_x f(x) \in \mathbb{R}^n$ and a symmetric matrix $H_f(x) \in \mathbb{R}^{n \times n}$ such that for all $y \in D$ we have

$$\begin{aligned} f(y) &= f(x) + \langle \nabla_x f(x), y - x \rangle + \frac{1}{2} \langle y - x, H_f(x)(y - x) \rangle \quad (9) \\ &\quad + \|y - x\|^2 R_2(y - x; f, x), \end{aligned}$$

where $\lim_{h \rightarrow 0} R_2(h; f, x) = 0$.

If $S \subset \text{int}(D)$, then we say f is twice differentiable at S (cz. dvakrát diferencovatelná v množině S), if it is twice differentiable at each $x \in S$.



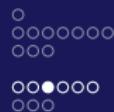
○○○○○

Druhá derivace - 2

Matrix $H_f(x)$ can differ from Hessian matrix. The reasons are

- ▶ $\nabla_x f$ does not exist in any neighborhood of x ,
- ▶ $\nabla_x f$ exists in a neighborhood of x and $\nabla_{x,x}^2 f(x)$ does not exist.
- ▶ $\nabla_x f$ exists in a neighborhood of x , $\nabla_{x,x}^2 f(x)$ exist, but, asymmetric.

Let us note the difference from Hessian is not mentioned in [1].



Druhá derivace - 3

Lemma 18

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$ and $x \in \text{int}(D)$. If f is twice differentiable at x then matrix $H_f(x)$ is uniquely determined.

Důkaz: Since $H_f(x)$ is symmetric, its uniqueness follows an observation on quadratic forms from linear algebra.

Q.E.D.



Druhá derivace - 4

Lemma 19

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$ and $x \in \text{int}(D)$. If f is differentiable at a neighborhood of x and $\nabla_x f$ is differentiable at x , then, $\nabla_{x,x}^2 f(x)$ exists and f is twice differentiable at x with

$$H_f(x) = \frac{1}{2} \nabla_{x,x}^2 f(x) + \frac{1}{2} (\nabla_{x,x}^2 f(x))^\top.$$

If, moreover, Hessian matrix is symmetric, i.e. $\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$ for all $i, j \in \{1, 2, \dots, n\}$, then

$$H_f(x) = \nabla_{x,x}^2 f(x).$$



Druhá derivace - 4hint

Důkaz: According to our assumptions, there is $\delta > 0$ such that $\mathcal{U}(x, \delta) \subset D$ and for all $y \in \mathcal{U}(x, \delta)$, $h \in \mathbb{R}^n$, $\|h\| < \delta - \|y - x\|$ we have

$$f(y + h) - f(y) = \langle \nabla_x f(y), h \rangle + \|h\| R_1(h; f, y),$$

$$\nabla_x f(y) - \nabla_x f(x) = \langle (\nabla_{x,x}^2 f(x))^\top, y - x \rangle + \|y - x\| R_1(y - x; \nabla_x f, x)$$

According to Lemma 10

$$f(x + h) - f(x) - \langle \nabla_x f(x), h \rangle = \int_0^1 \langle \nabla_x f(x + th) - \nabla_x f(x), h \rangle dt.$$

Plugging in expansion of gradient, we are receiving the statement.

Q.E.D.



Druhá derivace - 5

Lemma 20

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$, $x \in \text{int}(D)$ and $h \in \mathbb{R}^n$.

(i) If f is twice differentiable at x , then

$$\lim_{t \rightarrow 0} \frac{f(x + th) - f(x) - t \langle \nabla_x f(x), h \rangle}{t^2} = \frac{1}{2} \langle h, H_f(x) h \rangle.$$

(ii) Let us denote $D_h = \{t \in \mathbb{R} : x + th \in D\}$. If f is differentiable at a neighborhood of x and $\nabla_x f$ is differentiable at x , then, $\nabla_{x,x}^2 f(x)$ exists and function $\varphi : D_h \rightarrow \mathbb{R} : t \in D_h \rightarrow f(x + th)$ possesses derivatives

$$\begin{aligned}\varphi'(t) &= \langle \nabla_x f(x + th), h \rangle \quad \text{for all } t \text{ small enough,} \\ \varphi''(0) &= \langle h, \nabla_{x,x}^2 f(x) h \rangle.\end{aligned}$$



Matematická analýza - 0

Existence and continuity of gradient, resp. of Hessian, are sufficient conditions for differentiability in the sense of Definitions 6 and 17.

Lemma 21

Let $I \subset \mathbb{R}$, $\text{int}(I) \neq \emptyset$, $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $g : I \rightarrow D$, $f : D \rightarrow \mathbb{R}$ and $t \in \text{int}(I)$ such that $g(t) \in \text{int}(D)$. If gradient of f exists on a neighborhood of $g(t)$ and is continuous at $g(t)$ and g is differentiable at t , then $f \circ g$ is differentiable at t with

$$(f \circ g)'(t) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(g(t)) g'_i(t) = \langle \nabla_x f(g(t)), g'(t) \rangle.$$



○○○○○○○

Matematická analýza - 1

Using Lemma 21, we derive differentiability of a function.

Lemma 22

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$ and $x \in \text{int}(D)$. If gradient of f exists on a neighborhood of x and is continuous at x , then f is differentiable at x with

$$f(x + h) = f(x) + \langle \nabla_x f(x), h \rangle + \|h\| R_1(h; f, x),$$

$$|R_1(h; f, x)| \leq \max \{ \|\nabla_x f(x + uh) - \nabla_x f(x)\| : 0 \leq u \leq 1 \}$$

if h is sufficiently small.



Matematická analýza - 2

Lemma 23

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$ and $x \in \text{int}(D)$. Then,

f is continuously differentiable at a neighborhood of x if and only if there is $\delta > 0$ such that $\nabla_x f$ exists at $\mathcal{U}(x, \delta)$ and is continuous at $\mathcal{U}(x, \delta)$.

Důkaz: A consequence of Lemma 22.

Q.E.D.



○○○○○○

○○○○○

Matematická analýza - 3

Lemma 24

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$, $f : D \rightarrow \mathbb{R}$ and $x \in \text{int}(D)$. If $\nabla_x f$, $\nabla_{x,x}^2 f$ exist on a neighborhood of x and $\nabla_{x,x}^2 f$ is continuous at x , then Hessian $\nabla_{x,x}^2 f(x)$ is a symmetric matrix and f is twice differentiable at x with

$$\begin{aligned} f(x+h) &= f(x) + \langle \nabla_x f(x), h \rangle + \frac{1}{2} \langle h, \nabla_{x,x}^2 f(x) h \rangle + \\ &\quad + \frac{1}{2} \|h\|^2 R_2(h; f, x), \end{aligned}$$

$$|R_2(h; f, x)| \leq \max \left\{ \|\nabla_{x,x}^2 f(x + uh) - \nabla_{x,x}^2 f(x)\| : 0 \leq u \leq 1 \right\}$$

if h sufficiently small. Moreover, $H_f(x) = \nabla_{x,x}^2 f(x)$.

Konvexní funkce - 0

Convexity of a function can be verified by means of functions of one variable.

Theorem 25

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$ be a convex set and $f : D \rightarrow \mathbb{R}$. Then, function f is convex if and only if functions $\varphi_{x,s} : D_{x,s} \rightarrow \mathbb{R}$ are convex for all $x \in D$ and all $s \in \mathbb{R}^n$, where $\varphi_{x,s}(t) = f(x + ts)$ and

$D_{x,s} = \{t : x + ts \in D, t \in \mathbb{R}\}$. (Let us recall set $D_{x,s}$ is always an interval.)



Konvexní funkce - 2

Theorem 26

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$ be a convex open set and $f : D \rightarrow \mathbb{R}$ be differentiable at D . Then,

$$f \text{ is convex} \Leftrightarrow t \in D_{x,s} \mapsto \langle \nabla_x f(x + ts), s \rangle \text{ is nondecreasing on } D_{x,s} \text{ for all } x \in D, s \in \mathbb{R}^n. \quad (10)$$



Konvexní funkce - 3

Theorem 27

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$ be a convex open set and $f : D \rightarrow \mathbb{R}$. If f is differentiable at D and $\nabla_x f$ is differentiable at D , then, $\nabla_{x,x}^2 f$ exists on D , f is twice differentiable at D with

$$H_f(x) = \frac{1}{2} \nabla_{x,x}^2 f(x) + \frac{1}{2} (\nabla_{x,x}^2 f(x))^\top$$

and

$$f \text{ is convex} \Leftrightarrow H_f(x) \text{ is positively semidefinite for all } x \in D. \quad (11)$$



Konvexní funkce - 4

Definition 28

Let $D \subset \mathbb{R}^n$, $D \neq \emptyset$ be a set and $f : D \rightarrow \mathbb{R}$ be a function. We say, f possesses at $x \in D$ subgradient $a \in \mathbb{R}^n$ (cz. subgradient), if we have

$$f(y) - f(x) \geq \langle a, y - x \rangle \text{ for all } y \in D. \quad (12)$$

Set of all subgradients at x will be called subdifferential of f at x (cz. subdiferenciál) and will be denoted by $\partial f(x)$.



Konvexní funkce - 5

Subgradient and subdifferential are helpful tools for describing local minima of a convex function.

Lemma 29

Let $\mathcal{G} \subset \mathbb{R}^n$ be a nonempty open convex set, $f : \mathcal{G} \rightarrow \mathbb{R}$ be a convex function and $y \in \mathcal{G}$. Hence, the following is equivalent:

1. f is differentiable at y and $\partial f(y) = \{\nabla_x f(y)\}$.
2. $\partial f(y)$ is an one-point set.
3. f possesses a gradient at y .



Convex functions

○○○○●○

Konvexní funkce - 6

Results on separation of convex bodies have consequences for convex function.

Theorem 30

Let $D \subset \mathbb{R}^n$ be a nonempty convex set and $f : D \rightarrow \mathbb{R}$ be a convex function. Then, $\partial f(x) \neq \emptyset$ for each $x \in \text{rint}(D)$.

Equivalent description of a convex function using non-emptiness of subdifferentials is in power if function definition region is an open set.

Theorem 31

Let $D \subset \mathbb{R}^n$ be an open convex set and $f : D \rightarrow \mathbb{R}$. Then, f is a convex function if and only if $\partial f(x) \neq \emptyset$ for each $x \in D$.



oooooo●

Konvexní funkce - 7

For a continuous function, the characterization is also in power.

Theorem 32

Let $D \subset \mathbb{R}^n$ be a convex set and $f : D \rightarrow \mathbb{R}$ be a continuous function. Then, f is a convex function if and only if $\partial f(x) \neq \emptyset$ for each $x \in \text{rint}(D)$.

○
○○○○○○○
○○○
○○○○○○
○○○

○○○○○

Kužele - 0

Definition 33

Let $K \subset \mathbb{R}^n$ be a cone. We define polar of K (cz. polára K)

$$K^\circ = \{v \in \mathbb{R}^n : \forall x \in K \text{ we have } \langle v, x \rangle \leq 0\}. \quad (13)$$

and bipolar of K (cz. bipolára K)

$$K^{oo} = K^{\circ\circ} = \{w \in \mathbb{R}^n : \forall v \in K^\circ \text{ we have } \langle w, v \rangle \leq 0\}. \quad (14)$$

○
○○○○○○○
○○○
○○○○○○
○○○

○○○○○

Kužele - 1

Basic properties of polar.

Lemma 34

If $K \subset \mathbb{R}^n$ is cone, then K° is a closed convex cone and $K^{\circ\circ} = \text{clo}(\text{conv}(K))$.

○
 ○○○○○○○
 ○○○
 ○○○○○○
 ○○○

○○○○○

Kužele - 2

Definition 35

Let $M \subset \mathbb{R}^n$, $\tilde{x} \in \text{clo}(M)$. We define Tangent Cone to M at \tilde{x} (or, Cone of Tangents) (cz. tečný kužel k množině M v bodě \tilde{x}) by

$$T_M(\tilde{x}) = \left\{ s \in \mathbb{R}^n : \begin{array}{l} \exists x_k \in M, \lambda_k > 0 \ k \in \mathbb{N} \\ \text{s.t. } x_k \rightarrow \tilde{x}, \lambda_k(x_k - \tilde{x}) \rightarrow s. \end{array} \right\}. \quad (15)$$



○○○○○

Kužele - 3

Lemma 36

If $M \subset \mathbb{R}^n$, $\tilde{x} \in \text{clo}(M)$, then $T_M(\tilde{x})$ is a closed cone.

Lemma 37

If $M \subset \mathbb{R}^n$ is a convex set and $\tilde{x} \in \text{clo}(M)$, then $T_M(\tilde{x})$ is a closed convex cone.

Lemma 38

Let $M \subset \mathbb{R}^n$, $x \in \text{clo}(M)$ and $S \subset \mathbb{R}^n$, $x \in \text{int}(S)$. Then,

$$T_{M \cap S}(x) = T_{\text{clo}(M) \cap \text{clo}(S)}(x) = T_M(x) = T_{\text{clo}(M)}(x).$$



Kužele - 4

Definition 39

Let $S \subset \mathbb{R}^n$, $\tilde{x} \in \text{clo}(S)$. We say, that $s \in \mathbb{R}^n$ is
a Regular Normal to S at \tilde{x} (or, Normal to S at \tilde{x} in the Regular Sense),
(cz. regulární normála k množině S v \tilde{x}) if

$$\forall x \in S \text{ we have } \langle s, x - \tilde{x} \rangle \leq \|x - \tilde{x}\| R(x - \tilde{x}; s, \tilde{x}), \quad (16)$$

where $R(x - \tilde{x}; s, \tilde{x}) \rightarrow 0$ provided $x \rightarrow \tilde{x}$ and $x \in S$.

Regular Normal cone to S at \tilde{x} (or, Cone of Regular Normals to S at \tilde{x})
(cz. regulární normálový kužel)

$\hat{N}_S(\tilde{x})$ is a set of all regular normals to S at \tilde{x} .

○
○○○○○○○
○○○
○○○○○○
○○○

○○○○○

Kužele - 5

Definition 40

Let $S \subset \mathbb{R}^n$, $\tilde{x} \in \text{clo}(S)$. We say, that $s \in \mathbb{R}^n$ is a Normal to S at \tilde{x} (or, Normal to S at \tilde{x} in the General Sense; Normal Vector to S at \tilde{x}), (cz. normála k množině S v \tilde{x}) if there are sequences $x_k \in S$, $s_k \in \widehat{N}_S(x_k)$ for $k \in \mathbb{N}$ such that $x_k \rightarrow \tilde{x}$, $s_k \rightarrow s$.

Normal cone to S at \tilde{x} (or, Cone of Normals to S at \tilde{x}), (cz. Normálový kužel k množině S v bodě \tilde{x})

$N_S(\tilde{x})$ is the set of all normals to S at \tilde{x} .



○○○○○

Kužele - 6

Perceive defined objects are really cones and normal cone always contains regular normal cone.

Lemma 41

If $S \subset \mathbb{R}^n$ and $\tilde{x} \in \text{clo}(S)$, then $\widehat{N}_S(\tilde{x})$, $N_S(\tilde{x})$ are cones and $\widehat{N}_S(\tilde{x}) \subset N_S(\tilde{x})$.

Lemma 42

Let $M \subset \mathbb{R}^n$, $x \in \text{clo}(M)$ and $S \subset \mathbb{R}^n$, $x \in \text{int}(S)$. Then,

$$\begin{aligned}\widehat{N}_{M \cap S}(x) &= \widehat{N}_{\text{clo}(M) \cap \text{clo}(S)}(x) = \widehat{N}_M(x) = \widehat{N}_{\text{clo}(M)}(x) \text{ and} \\ N_{M \cap S}(x) &= N_{\text{clo}(M) \cap \text{clo}(S)}(x) = N_M(x) = N_{\text{clo}(M)}(x).\end{aligned}$$

○
○○○○○○○
○○○
○○○○○○
○○○

○○○○○

Kužele - 7

Theorem 43

If $S \subset \mathbb{R}^n$ and $\tilde{x} \in \text{clo}(S)$, then $T_S(\tilde{x})^\circ = \hat{N}_S(\tilde{x})$, $\hat{N}_S(\tilde{x})^\circ \supset T_S(\tilde{x})$.

○
○○○○○○○
○○○
○○○○○○
○○○

○○○○○

Kužele - 8

Polar of a normal cone has also certain importance.

Definition 44

For $S \subset \mathbb{R}^n$ and $\tilde{x} \in \text{clo}(S)$ we define Regular Tangent cone to S at \tilde{x}
(or, Cone of Regular Tangent Vectors of S at \tilde{x}) (cz. regulární tečný
kužel k množině S v bodě \tilde{x}) by

$$\hat{T}_S(\tilde{x}) = \left\{ s \in \mathbb{R}^n : \begin{array}{l} \text{for each } x_k \in S, k \in \mathbb{N}, x_k \rightarrow \tilde{x}, \\ \text{for each } \lambda_k > 0, k \in \mathbb{N}, \lambda_k \nearrow +\infty, \\ \text{there is } \xi_k \in S, k \in \mathbb{N}, \\ \text{such that } \xi_k \rightarrow \tilde{x}, \lambda_k(\xi_k - x_k) \rightarrow s. \end{array} \right\}. \quad (17)$$



○○○○○○

Kužele - 9

At first, consider basic properties of a regular tangent cone.

Theorem 45

If $S \subset \mathbb{R}^n$ and $\tilde{x} \in \text{clo}(S)$, then $\widehat{T}_S(\tilde{x})$ is a closed convex cone.

Lemma 46

Let $M \subset \mathbb{R}^n$, $x \in \text{clo}(M)$ and $S \subset \mathbb{R}^n$, $x \in \text{int}(S)$. Then,

$$\widehat{T}_{M \cap S}(x) = \widehat{T}_{\text{clo}(M) \cap \text{clo}(S)}(x) = \widehat{T}_M(x) = \widehat{T}_{\text{clo}(M)}(x).$$

Theorem 47

If $S \subset \mathbb{R}^n$ and $\tilde{x} \in \text{clo}(S)$, then $\widehat{T}_S(\tilde{x}) \subset T_S(\tilde{x})$.



○○○○○

Kužele - lokální uzavřenost

Definition 48

Let $S \subset \mathbb{R}^n$, $\tilde{x} \in \text{clo}(S)$. We say, that set S is locally closed at \tilde{x} (cz. lokálně uzavřená v \tilde{x}), if there is $\delta > 0$ such that $\mathcal{V}(\tilde{x}, \delta) \cap S$ is a closed set.

○
○○○○○○○
○○○
○○○○○○
○○○

○○○○○

Kužele - 10

Theorem 49

Let $S \subset \mathbb{R}^n$ and $\tilde{x} \in \text{clo}(S)$. If S is locally closed at \tilde{x} , then

$$\widehat{T}_S(\tilde{x}) = \left\{ s \in \mathbb{R}^n : \begin{array}{l} \text{For each } x_k \in S, k \in \mathbb{N}, x_k \rightarrow \tilde{x}, \\ \text{there are } s_k \in T_S(x_k), k \in \mathbb{N} \\ \text{such that } s_k \rightarrow s. \end{array} \right\} \quad (18)$$

○
○○○○○○○
○○○
○○○○○○
○○○

○○○○○

○
○○○○○○○
○○○
○○○○○○
○○○

○○○○○

Kužele - 12

Definition 51

Let $S \subset \mathbb{R}^n$, $\tilde{x} \in S$. We say, S is regular at \tilde{x} in the Sense of Clarke, (cz. regulární ve smyslu Clarka), if S is locally closed at \tilde{x} and $N_S(\tilde{x}) = \hat{N}_S(\tilde{x})$.

○
○○○○○○○
○○○
○○○○○○
○○○

○○○○○

Kužele - 13

Lemma 52

Let $S \subset \mathbb{R}^n$ be convex, $\tilde{x} \in S$. Then,

$$T_S(\tilde{x}) = \text{clo}(\{s \in \mathbb{R}^n : \exists \lambda > 0 \text{ such that } \tilde{x} + \lambda s \in S\}),$$

$$\text{int}(T_S(\tilde{x})) = \{s \in \mathbb{R}^n : \exists \lambda > 0 \text{ such that } \tilde{x} + \lambda s \in \text{int}(S)\},$$

$$N_S(\tilde{x}) = \widehat{N}_S(\tilde{x}) = \{s \in \mathbb{R}^n : \forall x \in S \text{ we have } \langle s, x - \tilde{x} \rangle \leq 0\}.$$

Therefore, convex set S is regular at \tilde{x} in sense of Clarke if and only if S is locally closed at \tilde{x} .

O
○○○○○○○
○○○
○○○○○○
○○○

○○○○○

Reference

-  Bazara, M.S.; Sherali, H.D.; Shetty, C.M.: *Nonlinear Programming. Theory and Algorithms.* 2nd edition, Wiley, New York, 1993.
-  Rockafellar, T.: *Convex Analysis.*, Springer-Verlag, Berlin, 1975.
-  Rockafellar, T.; Wets, R. J.-B.: *Variational Analysis.* Springer-Verlag, Berlin, 1998.