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# INDEPENDENCE PROOFS AND COMBINATORICS

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ABSTRACT. This is an exposition, for non-logicians, of some applications of combinatorial set theory in the theory of forcing and generic sets. It is argued that independence questions in logic are fundamentally combinatorial, and that forcing is simply a translation process for converting such independence questions into combinatorial propositions that can be proved outright.

INTRODUCTION. One of the most fruitful areas for the application of combinatorics in mathematical logic has been the theory of forcing and generic sets. As with most of the other areas of logic, the combinatorial applications in forcing have been from that branch of the subject known as infinitary combinatorics or combinatorial set theory.

What makes the applications of combinatorics to forcing interesting is that forcing is itself a tool that can be applied to problems in many different areas, both inside and outside logic. In particular, forcing can be applied to problems in combinatorial set theory. Thus we shall be talking about combinatorics as applied to forcing, as applied to combinatorics!

The thesis of the paper is quite easily set down. It is simply that forcing is a translation technique for converting combinatorial propositions into combinatorial propositions. Given a proposition P, the theory of forcing shows how to convert a question about the consistency of P into a question about the provability (or, if you like, the truth) of another proposition Q. Thus, proving Q will show that P is consistent. Moreover, if P is combinatorial, so will be Q. Several examples of this phenomenon are given in §§5-9, and the translation process is explicitly exhibited in §8.

Since most of this symposium has been concerned with finite combinatorics, we have tried to make this paper comprehensible to anyone familiar with that subject. The examples from combinatorial set theory involve intersection properties of sets and the partition calculus, both of which have interesting finite analogues. Complete proofs are given (in §§2 and 3) of all necessary combinatorial results.

Forcing is introduced in §4 by means of eight axioms, which are

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accompanied by an interpretation in intuitive terms. The exposition is complete enough to handle most elementary forcing proofs but (we hope) simple enough to be understood by a reader who has never seen the notion before.

§§5-7 are devoted to a proof of the consistency of the partition relation $\aleph_2 \not\to [\aleph_1]^2_{\aleph_0}$. This result is part of the folklore of combinatorial set theory. The expert will see at once that the partial ordering used for this result is just Cohen's original ordering for showing the consistency of $2^{\aleph_0} \geq \aleph_2$, but in a slightly different form.

§§8 and 9 are concerned with problems about almost disjoint sets. The paper ends with the proof of a special case of a result in [1].

The reader interested in seeing more combinatorial set theory should consult [8].

We use standard set-theoretical notation throughout. If $A$ is a set then $|A|$ is the cardinality of $A$ and $[A]^n$ is the set of n-element subsets of $A$.

We assume that ordinal numbers have been defined so that each ordinal coincides with the set of its predecessors. Thus $\alpha \in \beta$ and $\alpha < \beta$ are synonymous for ordinals. Since the axiom of choice is assumed, every set is equipotent with some ordinal number. The least ordinal of cardinality $\aleph_\alpha$ is denoted by $\omega_\alpha$, and $\omega_0 = \omega$. Thus $\omega$ is the set of all nonnegative integers, and $\omega_1$ is the set of all countable (or finite) ordinal numbers. The purpose of this approach is to guarantee that $\omega_\alpha$ is a canonical set of cardinality $\aleph_\alpha$. Usually $\aleph_\alpha$ is defined so that $\aleph_\alpha = \omega_\alpha$, but this fact plays no role in the paper.

2. INTERSECTION PROPERTIES OF SETS. A family $F$ of sets is called a $\Delta$-system iff there is a set $\Delta$ (called the <u>kernel</u> of the $\Delta$-system) such that $A \cap B = \Delta$ whenever $A, B \in F$ and $A \neq B$.

Erdős and Rado [3] have made an extensive study of $\Delta$-systems; all we shall need here, however, is the following theorem, due to Mazur [6].

THEOREM 2.1. (Mazur) If $F$ is an uncountable family of finite sets, then there is an uncountable $\Delta$-system $F' \subseteq F$.

PROOF. Without loss of generality, we may assume that all the elements of $F$ have cardinality $n$. The proof goes by induction on $n$. For $n = 1$ the result is trivial. Suppose $n = m + 1$. We consider two cases.

CASE 1. There is some $a$ such that $\{A \in F: a \in A\}$ is uncountable. Let $F_1 = \{A - \{a\}: a \in A, A \in F\}$. By inductive hypothesis there is an uncountable $\Delta$-system $F_2 \subseteq F_1$. But then $F' = \{B \cup \{a\}: B \in F_2\}$ is an uncountable $\Delta$-system contained in $F$.

CASE 2. For all $a$, $\{A \in F: a \in A\}$ is at most countable. Now we construct by transfinite induction a sequence $\langle A_\alpha: \alpha < \omega_1\rangle$ of elements of $F$ so that if $\alpha < \beta$ then $A_\alpha \cap A_\beta = 0$. Given $A_\beta$ for all $\beta < \alpha$, we know that $\bigcup\{A_\beta: \beta < \alpha\}$ is countable, so by the hypothesis for this case $\{A \in F: \exists\beta < \alpha\ A \cap A_\beta \neq 0\}$ is countable, so $A_\alpha$ may be chosen disjoint from all the $A_\beta$, $\beta < \alpha$.

Two sets $A$ and $B$ are said to be almost disjoint if $|A \cap B| < |A|, |B|$. For finite sets this simply says neither is included in the other, but for infinite sets it is quite an interesting notion. An infinite set $X$ may be decomposed into no more than $|X|$ disjoint sets, but what if the sets are only required to be almost disjoint?

THEOREM 2.2. (Sierpiński) There is a family $F$ of pairwise almost disjoint subsets of $\omega$ such that $|F| = 2^{\aleph_0}$.

PROOF. For each real number $x$, let $\langle x_n : n \in \omega \rangle$ be a sequence of rational numbers converging to $x$. Then the sets $\{x_n : n \in \omega\}$ are almost-disjoint. Since the rationals are countable and the reals have cardinality $2^{\aleph_0}$, this completes the proof.

THEOREM 2.3. (Sierpiński-Tarski) Assume $2^{\aleph_0} = \aleph_1$. Then there is a family $F$ of subsets of $\omega_1$ such that $|F| = 2^{\aleph_1}$, $|A| = \aleph_1$ for all $A \in F$, and $|A \cap B| \leq \aleph_0$ whenever $A, B \in F$ and $A \neq B$.

PROOF. It is possible to give a proof similar to the one above, but the following is a little more direct. Let $X$ be the set of all functions mapping $\omega_1$ into $\{0,1\}$. Then $|X| = 2^{\aleph_1}$. If $f \in X$ and $\alpha < \omega_1$ let $f|\alpha$ denote the restriction of $f$ to $\alpha$ $(=\{\beta : \beta < \alpha\})$. For each $f$, let $A_f = \{f|\alpha : \alpha < \omega_1\}$. Then if $f \neq g$, $|A_f \cap A_g| \leq \aleph_0$. Also, $|A_f| = \aleph_1$. Finally, it will suffice to observe that $\{f|\alpha : f \in X, \alpha < \omega_1\}$ has cardinality $2^{\aleph_0} = \aleph_1$. But this is clear.

It is natural to ask whether the continuum hypothesis can be eliminated from Theorem 2.3. It is shown in [1] that it can be replaced by a much weaker cardinality assumption, but that it cannot be eliminated altogether. The latter result will occupy us in §9.

3. THE PARTITION CALCULUS. We shall be interested in two kinds of partition relations.

Let $\kappa$, $\lambda$, and $\mu$ be cardinal numbers (possibly finite), and let $n$ be a positive integer. If $A$ is a set then $[A]^n$ denotes the set of $n$-element subsets of $A$.

The symbol $\kappa \to (\lambda)^n_\mu$ means that if $|A| = \kappa$, $|C| = \mu$, and $f: [A]^n \to C$, then there must exist $B \subseteq A$ such that $|B| = \lambda$ and $f$ is constant on $[B]^n$. The set $B$ is said to be homogeneous for $f$.

The symbol $\kappa \to [\lambda]^n_\mu$ is defined exactly as above except that we only require that the range of $f$ on $[B]^n$ is not all of $C$. We say $B$ is weakly homogeneous for $f$.

In this notation Ramsey's Theorem may be expressed as

$$\aleph_0 \to (\aleph_0)^n_k \quad \text{for all finite } n \text{ and } k.$$

Since we will only be interested in the case $n = 2$, both these partition relations may be regarded as graph-coloring properties. For example, $\kappa \to [\lambda]^2_\mu$ says that if the edges of the complete graph on $\kappa$ vertices are colored with $\mu$ colors then there is a complete subgraph of cardinality $\lambda$ which omits at least one of the colors.

We will need the following theorem, which is a special case of a theorem of Erdős and Rado [4].

THEOREM 3.1. Suppose $2^{\aleph_0} = \aleph_1$. Then $\aleph_2 \rightarrow (\aleph_1)^2_{\aleph_0}$.

PROOF. Partition relations of this form where $\kappa$, $\lambda$ and $\mu$ are positive integers are usually proved by constructing trees with sufficiently long branches, and this theorem is just the same. The only difference is that the tree is not finite in height.

Suppose $|A| = \aleph_2$ and $h\colon [A]^2 \rightarrow \omega$.

Let $F$ denote the set of all functions mapping $\omega_1$ into $\omega$. If $f \in F$ and $\alpha < \omega_1$, then $f|\alpha$ denotes the restriction of $f$ to $\alpha$ ($= \{\beta\colon \beta < \alpha\}$). The objects of the form $f|\alpha$ form a tree if we put $f|\beta$ above $f|\alpha$ whenever $\alpha < \beta$.

Let $f \in F$. By induction on $\alpha < \omega_1$ we will obtain $A(f|\alpha) \subseteq A$ and (if $A(f|\alpha) \neq 0$) an element $a(f|\alpha) \in A(f|\alpha)$.

Let $A(f|0) = A$.

If $\alpha$ is a limit ordinal, let $A(f|\alpha) = \bigcap\{A(f|\beta)\colon \beta < \alpha\}$.

If $\alpha = \beta + 1$, then let $A(f|\alpha) = \{a \in A(f|\beta)\colon a \neq a(f|\beta)$ and $h\{a, a(f|\beta)\} = f(\beta)\}$.

In each case, let $a(f|\alpha) \in A(f|\alpha)$ if possible.

It is easy to see that if $a \in A$ and $\alpha < \omega_1$, then either $a = a(f|\beta)$ for some $f \in F$ and $\beta < \alpha$ or else there is $f \in F$ such that $a \in A(f|\alpha)$. Note that if $f|\alpha \neq g|\alpha$ then $A(f|\alpha) \cap A(g|\alpha) = 0$.

Now there are only $2^{\aleph_0}$ elements of the form $a(f|\alpha)$, and since $2^{\aleph_0} = \aleph_1$ there must be $a \in A$ such that $a \neq a(f|\alpha)$ for all $f \in F$ and $\alpha < \omega_1$. Choose $g \in F$ so that $a \in A(g|\alpha)$ for all $\alpha < \omega_1$. Then $A(g|\alpha) \neq 0$ for all $\alpha$ so $a(g|\alpha)$ always exists. Denote $a(g|\alpha)$ by $a_\alpha$.

Notice that if $\alpha < \beta$, then since $a_\beta \in A(g|(\alpha+1))$ we have $h\{a_\alpha, a_\beta\} = g(\alpha)$. Since $g\colon \omega_1 \rightarrow \omega$ there is $k \in \omega$ such that $g^{-1}\{k\}$ is uncountable. But then $\{a_\alpha\colon \alpha \in g^{-1}\{k\}\}$ is homogeneous for $h$.

This argument can easily be generalized to larger cardinals.

We will not need any specific results about the "square-bracket" relation, but by way of orientation we mention the following:

(1) If $2^{\aleph_0} = \aleph_1$, then $\aleph_1 \not\rightarrow [\aleph_1]^2_{\aleph_1}$ (Erdős-Hajnal-Rado [2])

(2) $\aleph_1 \not\rightarrow [\aleph_1]^2_4$ (Galvin [5])

(3) $2^{\aleph_0} \not\rightarrow [2^{\aleph_0}]^2_{\aleph_0}$ (Shelah [5])

It is open whether $\aleph_1 \rightarrow [\aleph_1]^2_{\aleph_0}$ or $2^{\aleph_0} \rightarrow [\aleph_1]^2_3$ is consistent with the axioms of set theory.

In §7 we will show the consistency of $\aleph_2 \not\rightarrow [\aleph_1]^2_{\aleph_0}$.

## 4. FORCING.

Few recent developments in mathematics have had as much impact on Cohen's theory of forcing and generic sets. Almost overnight, literally hundreds of long-outstanding problems in set theory were solved. Nor have the applications of forcing been confined to set theory. The technique has been used in measure theory, topology, homological algebra and other areas. The surprising thing is that so few mathematicians are familiar with forcing. There seems to be a feeling that forcing is a mysterious technique which one must be a logician to understand.

Nothing could be farther from the truth. After all, Cohen wasn't a logician when he discovered forcing!

One of the remarkable things about forcing is that the proof that the technique works has almost nothing to do with the way it is applied. In an effort to dispel some of the mystery about it, therefore, we present here a fairly concise axiomatic treatment of forcing which is still complete enough for most applications. The reader who wants to see a proof of the axioms is welcome to track one down (see [7], for example), but should be warned that the time would be better spent in learning some of the applications of forcing.

The theory of forcing provides a systematic way of imagining a universe for set theory which is larger than the real universe. This is necessary to do in connection with problems such as whether the negation of the Continuum Hypothesis (CH) is consistent with the axioms of set theory. If CH is false in the (real) universe of set theory then nothing more need be done, but what if CH is true? Then it is necessary to imagine a possible world for set theory in which there are more than $\aleph_1$ real numbers, and forcing shows the way.

Forcing begins with a partial ordering $(P,\leq)$. We think of an element $p$ of $P$ as giving a certain amount of information or evidence about the imaginary universe of set theory (usually the information is about a certain object in the imaginary universe). We interpret $p \leq q$ as saying that $q$ gives more information than $p$. Let $\dot{V}$ denote the imaginary universe.

Let us suppose that we are equipped with a language[1] for talking about $\dot{V}$. Since all the real sets (belonging to the real universe) belong to $\dot{V}$, we will need symbols to denote them as well as symbols to denote the imaginary elements of $\dot{V}$. Our convention is as follows: If $x$ is a real set then we simply use $x$ to denote $x$; a symbol with a dot over it, like $\dot{f}$, denotes a (possibly) imaginary element. Now we are ready to talk about forcing. Here are the axioms, but before trying to understand them please read the explanation that comes afterward. There is a relation[2] $p \Vdash \varphi$ (read "$p$ forces $\varphi$" between certain $p \in P$ and certain assertions $\varphi$ about $\dot{V}$ such that the following are true:

AXIOM 1. $p \Vdash \varphi$ and $\psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$.

AXIOM 2. $p \Vdash$ not $\varphi$ iff $(\forall q \geq p) q \nVdash \varphi$ ($\nVdash$ means "does not force").

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[1] This is just the usual first-order language of set theory with added symbols as indicated. Nearly all mathematical statements can be expressed in this language, so we will usually speak mathematics instead of logic.

[2] Strictly speaking, this is not true. For each positive integer n there is a relation $p \Vdash_n \varphi$ as above which works as long as $\varphi$ has quantifier-depth $\leq n$. Since none of our forcing arguments (and very few of anybody else's) need $\varphi$'s of arbitrarily large quantifier-depth, we ignore this distinction.

AXIOM 3. If $A$ is a set then $p \Vdash (\exists x \in A)\varphi(x)$ iff $(\forall q \geq p)(\exists r \geq q)(\exists a \in A)\ r \Vdash \varphi(a)$.

AXIOM 4. If $p \Vdash \varphi$ and $q \geq p$ then $q \Vdash \varphi$.

AXIOM 5. If $\varphi$ implies $\psi$ and $p \Vdash \varphi$, then $p \Vdash \psi$.

AXIOM 6.[1] If $\varphi$ is one of the axioms of ZFC (Zermelo-Fraenkel set theory together with the axiom of choice), then $\forall p\ p \Vdash \varphi$.

The set-theorist approaching a forcing problem tends to think rather like a detective trying to solve a crime. In particular, $p \Vdash \varphi$ may be interpreted as saying that on the basis of the evidence $p$ we are forced to conclude that $\varphi$ is true.

With this interpretation, Axiom 1 is clear. Axiom 2 says, quite reasonably, that we can only conclude that $\varphi$ is false if no conceivable further evidence could possibly convince us that $\varphi$ is true.

Axiom 3 can be treated similarly. One is tempted to suppose that $p \Vdash (\exists x \in A)\varphi(x)$ iff $\exists a \in A\ p \Vdash \varphi(a)$, but a moment's reflection shows that this cannot be true. Sherlock Holmes could certainly conclude on the basis of a piece of evidence that one of the people in the room committed the murder, even if he didn't know which one did it. All that is required is that no conceivable further evidence should exonerate all the suspects. And that is exactly what Axiom 3 says.

The rest of the axioms are self-explanatory. Of course Axiom 6 is a very remarkable and deep fact, and we do not mean to suggest that it is obvious.

The reader may find it instructive to deduce the following from the axioms and justify it detective-theoretically:

$$p \Vdash \varphi \text{ or } \psi \quad \text{iff} \quad (\forall q \geq p)(\exists r \geq q)\ r \Vdash \varphi \text{ or } r \Vdash \psi.$$

(Hint: disjunction is definable in terms of conjunction and negation.) As another exercise, try:

$$p \Vdash (\forall x \in A)\varphi(x) \quad \text{iff} \quad \forall a \in A\ p \Vdash \varphi(a).$$

Now suppose $\forall p\ p \Vdash \varphi$. We assert that (if ZFC is consistent) $\varphi$ is consistent with ZFC. If not, then not $\varphi$ is implied by finitely many axioms of ZFC, so by Axioms 5 and 6, $\forall p\ p \Vdash$ not $\varphi$, and this contradicts Axiom 2. The consistency of ZFC is required because Axioms 1-6 are proved in ZFC.

Hence to show the consistency of some proposition $\varphi$ with ZFC, we need only find some partial ordering $P$ such that $(\forall p \in P)\ p \Vdash \varphi$.

The reader who is meeting with forcing for the first time may now find it convenient to skip to the next section, where an example is given which should make the notion a little clearer.

The rest of this section is devoted to a discussion of the kinds of imaginary sets that can occur in $\dot{V}$.

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[1]Since there are axioms of ZFC of arbitrary large quantifier-depth, this should really be replaced by infinitely many axioms (one for each $\varphi$, for instance) but this is a minor point.

It turns out that $\dot{V}$ is generated by a very important imaginary set $\dot{G}$, which we can think of as the set of all true or correct information in P.

Let us say that p and q are compatible if $(\exists r \in P) p, q \leq r$; otherwise p and q are incompatible. We can think of incompatible p and q as giving conflicting information about $\dot{V}$.

Let us say that a set $G \subseteq P$ is generic if

(1) if $p \in G$ and $q \leq p$ then $q \in G$.

(2) if $p, q \in G$ then p and q are compatible.

(3) for any $\varphi$ there is $p \in G$ such that $p \Vdash \varphi$ or $p \Vdash$ not $\varphi$.

Thus G consists of compatible information, and by (3) G gives us complete information about $\dot{V}$. It should not come as a surprise, then, that $\dot{V}$ is generated by a generic set.

AXIOM 7. There is a symbol $\dot{G}$ such that $\forall p\ p \Vdash$ "$\dot{G}$ is generic and $p \in \dot{G}$".

Note that $p \Vdash p \in \dot{G}$ simply says that on the basis of the information p, we conclude that p is correct.

AXIOM 8. $\forall p\ p \Vdash$ "every element of $\dot{V}$ is definable from $\dot{G}$ together with finitely many real sets".

The careful reader will object that statement (3) in the definition of generic is illegal since, contrary to the proviso in an earlier footnote, the quantifier-depth of $\varphi$ is not bounded. There is a way around this difficulty, but since it results in a less transparent statement we have reserved it for the end of this section so that the reader who is not bothered by such things can simply skip it.

Call a set $D \subseteq P$ dense in P iff $(\forall p \in P)(\exists q \geq p) q \in D$. Now replace (3) by

(3') If D is a (real) set and D is dense in P, then $D \cap G \neq 0$.

Note that (3') implies (3) since for any $\varphi$ $\{p\colon p \Vdash \varphi$ or $p \Vdash$ not $\varphi\}$ is dense in P (use Axiom 2).

The reader who prefers to think in terms of models of set theory may find the imaginary universe $\dot{V}$ uncongenial. If that should happen then let us remark that under certain circumstances the imaginary can be made real. If M is a countable model of set theory and $(P, \leq) \in M$ then there is a set $G \subseteq P$ satisfying (1), (2), and (3') for all $D \in M$, and there is a model N of ZFC such that $G \in N$ and for all $\varphi$, $\varphi$ is true in N iff $\exists p \in G\ p \Vdash \varphi$ (where the interpretation of $\dot{G}$ in N is G and other terms are similarly interpreted).

5. AN EXAMPLE. Let us consider the problem of showing the consistency of the partition relation $\aleph_2 \not\to [\aleph_1]^2_{\aleph_0}$. Note that this is a nontrivial problem since, by Theorem 3.1, if $\aleph_2 \not\to (\aleph_1)^2_{\aleph_0}$ then $2^{\aleph_0} \geq \aleph_2$.

Recall that $\aleph_2 \not\to [\aleph_1]^2_{\aleph_0}$ means that if $|A_0| = \aleph_2$ then there is

$F: [A_0]^2 \to \omega$ such that for any $B \subseteq A_0$, if $|B| = \aleph_1$ then the range of $F$ on $[B]^2$ is $\omega$. Let us keep $A_0$ fixed for the rest of the paper.

If we try to construct an imaginary universe $\dot{V}$ in which $\aleph_2 \not\to [\aleph_1]^2_{\aleph_0}$ is true, then we must make sure that there is a function $\dot{F}$ in $\dot{V}$ as above (of course $\dot{F}$ will probably be imaginary). Thus a natural partial ordering to consider is one whose elements give information about $\dot{F}$. One such ordering is the set $P$ of all functions $p$ such that for some finite subset $X$ of $A_0$, $p: [X^2] \to \omega$. Put $p \leq q$ iff $p \subseteq q$ (i.e., iff domain(p) $\subseteq$ domain(q) and $q$ agrees with $p$ on domain(p)). We interpret $p$ as telling us the values of $\dot{F}$ on $[X]^2$.

FACT. There is $\dot{F}$ such that $\forall p\ p \Vdash$ "$\dot{F}: [A_0]^2 \to \omega$ and $p \in \dot{F}$".

Simply let $\dot{F}$ be $\bigcup\{p: p \in \dot{G}\}$ and apply Axiom 7. The details are left to the reader.

Now we prove $\forall p\ p \Vdash \aleph_2 \not\to [\aleph_1]^2_{\aleph_0}$.

6. AVOIDING CARDINAL COLLAPSE. Suppose $A$ and $B$ are infinite sets. Then to say that $|A| \neq |B|$ is to say that there is no one-to-one function mapping $A$ onto $B$. Now, the nonexistence of such a function is really more a deficiency in the universe of set theory than it is an intrinsic property of the sets $A$ and $B$. In particular, it is quite possible that in imagining the universe $\dot{V}$ we may inadvertently imagine the existence of a one-to-one correspondence $\dot{f}$ between $A$ and $B$. This phenomenon is known as <u>cardinal collapse</u>. Sometimes it is induced deliberately, but in our case it would be a disaster. If, for example, $A_0$ became a countable set in $\dot{V}$ then $\dot{F}$ would be utterly useless for proving that $\aleph_2 \not\to [\aleph_1]^2_{\aleph_0}$. Thus we must show that for the $P$ of §5, cardinal collapse does not occur.

Let us remark that cardinal collapse never occurs for finite sets. No amount of imagining could ever produce a one-to-one correspondence between a set with no element and a set with one element.

A partial ordering $(P,\leq)$ is said to have the <u>countable chain condition</u> (the c.c.c.) iff every set of pairwise incompatible elements is countable (or finite).

THEOREM 6.1. If $(P,\leq)$ has the c.c.c. and $A$ and $B$ are infinite sets such that $|A| < |B|$, then $\forall p \in P\ p \Vdash |A| < |B|$.

PROOF. It will suffice to show that if $p \Vdash \dot{f}: A \to B$, then $p \Vdash$ "the range of $\dot{f}$ is not all of $B$".

Suppose $q_1, q_2 \geq p$, $a \in A$, $b_1, b_2 \in B$, $b_1 \neq b_2$, and $q_1 \Vdash \dot{f}(a) = b_1$ and $q_2 \Vdash \dot{f}(a) = b_2$. Then $q_1$ and $q_2$ must be incompatible, since if $r \geq q_1, q_2$ we would have by Axiom 4 $r \Vdash \dot{f}(a) = b_1$ and $\dot{f}(a) = b_2$, a contradiction. Hence $\{b: \exists q \geq p\ q \Vdash \dot{f}(a) = b\}$ must be countable since $P$ has the c.c.c. It follows immediately that if $Y = \{b: (\exists q \geq p)(\exists a \in A) q \Vdash \dot{f}(a) = b\}$, then $|Y| = |A|$. Hence there is $b \in B - Y$. We claim $p \Vdash b \notin$ range $\dot{f}$, and this will complete the proof.

If $p \not\Vdash b \in$ range $\dot{f}$, then by Axiom 2 there is $q \geq p$ such that $q \Vdash b \in$ range $\dot{f}$, i.e., $q \Vdash (\exists a \in A) \dot{f}(a) = b$. Hence by Axiom 3 there is $r \geq q$

and $a \in A$ such that $r \Vdash \dot{f}(a) = b$. But then $b \in Y$, which is impossible.

It remains only to show the following:

THEOREM 6.2. The partial ordering $(P,\leq)$ of §5 has the c.c.c.

PROOF. Suppose, on the contrary, that $A$ is an uncountable pairwise incompatible subset of $P$. For each $p \in P$, domain(p) is a finite subset of $[A_0]^2$, so by Theorem 2.1 there is uncountable $B \subseteq A$ such that $\{\text{domain}(p): p \in B\}$ is a $\Delta$-system with kernel $\Delta$. If $p \in B$ then $p|\Delta$ (the restriction of $p$ to $\Delta$) maps $\Delta$ into $\omega$. Since $B$ is uncountable there are $p, q \in B$ such that $p \neq q$ but $p|\Delta = q|\Delta$. But since Domain(p) $\cap$ domain (q) $= \Delta$, $p \cup q$ is a function, and hence there is $r \in P$ such that $r \geq p, q$. Thus $p$ and $q$ are compatible, contradiction.

Notice the purely combinatorial nature of this argument.

## 7. $\aleph_2 \not\to [\aleph_1]^2_{\aleph_0}$.

Throughout this section $(P,\leq)$ is the partial ordering of §5.

THEOREM 7.1. $(\forall p \in P)\ p \Vdash \aleph_2 \not\to [\aleph_1]^2_{\aleph_0}$.

PROOF. In view of the results of the previous section it will suffice to prove that if $n \in \omega$ and

$$p \Vdash \dot{A} \subseteq A_0 \quad \text{and} \quad |\dot{A}| = \aleph_1,$$

then

(*) $p \Vdash n$ **belongs** to the range of $\dot{F}$ on $[\dot{A}]^2$.

Suppose (*) is false. Then by Axiom 2 there is $p' \geq p$ such that

(**) $p' \Vdash n$ does not belong to the range of $\dot{F}$ on $[\dot{A}]^2$.

Let $B = \{a \in A_0: \exists q \geq p'\ q \Vdash a \in \dot{A}\}$. We claim that $p' \Vdash \dot{A} \subseteq B$. If not, then by Axiom 2 there is $q \geq p'$ such that $q \Vdash (\exists a \in A_0 - B)\ a \in \dot{A}$. By Axiom 3 there is $a \in A_0 - B$ and $r \geq q$ such that $r \Vdash a \in \dot{A}$. But then $a \in B$, contradiction. Hence $p' \Vdash \dot{A} \subseteq B$. It follows that $B$ must be uncountable.

For each $a \in B$ choose $p_a \geq p'$ such that $p_a \Vdash a \in \dot{A}$. Let $X_a$ be a finite subset of $A_0$ such that $p_a: [X_a]^2 \to \omega$. We may assume $a \in X_a$ since otherwise we could enlarge $p_a$ by adding $a$ to $X_a$ and by Axiom 4 it would still be true that $p_a \Vdash a \in \dot{A}$.

Now by Theorem 2.1 there is uncountable $C \subseteq B$ such that $\{X_a: a \in C\}$ form a $\Delta$-system with kernel $\Delta$. Clearly there exist $a, b \in C$ such that $a, b \notin \Delta$, $a \neq b$, and $p_a|[\Delta]^2 = p_b|[\Delta]^2$. But then $a \notin X_b$ and $b \notin X_a$ so there is $r: [X_a \cup X_b]^2 \to \omega$ such that $r|[X_a]^2 = p_a$, $r|[X_b]^2 = p_b$, and $r\{a,b\} = n$. Since $r \Vdash r \subseteq \dot{F}$ (see the Fact in §5), we have $r \Vdash a, b \in \dot{A}$ and $\dot{F}\{a,b\} = n$. Hence $r \Vdash n$ belongs to the range of $\dot{F}$ on $[\dot{A}]^2$, contradicting (**) and the fact that $r \geq p'$.

Once again, note the combinatorial nature of the proof. Except for occasional fairly trivial appeals to the forcing axioms, the whole proof rests on Theorem 2.1. It is also interesting to note that there is nothing special about $\aleph_2$ in any of this. We could have assumed $|A_0| = \aleph_3$ and proved $\forall p\ p \Vdash \aleph_3 \not\to [\aleph_1]^2_{\aleph_0}$, and so forth.

## 8. ALMOST DISJOINT SETS REVISITED.

In §§5-7 we have seen an intersection property of sets (Theorem 2.1) used to obtain consistency results about partition relations. In §9 we shall reverse the process, using a partition relation to prove a consistency result about the maximum number of almost disjoint sets of a certain kind. Before we begin that argument, however, it will be helpful to see how combinatorial questions about almost disjoint sets in $\dot{V}$ can be translated into purely combinatorial questions about real sets in the universe of set theory.

Suppose $(P,\leq)$ is a partial ordering such that cardinal collapse does not occur under forcing with P. Then $\omega_1$, the set of all countable ordinals, is still the set of all countable ordinals in $\dot{V}$. Suppose $\forall p\ p \Vdash \dot{X} \subseteq \omega_1$. For each $\alpha \in \omega_1$, let $X_\alpha = \{p: p \Vdash \alpha \in \dot{X}\}$. Then even though $\dot{X}$ is imaginary, $\dot{X}$ is <u>represented</u> by the sequence $\langle X_\alpha: \alpha \in \omega_1\rangle$, which is a real set. It is not difficult to see that if $\langle X_\alpha: \alpha \in \omega_1\rangle = \langle Y_\alpha: \alpha \in \omega_1\rangle$ then $\forall p\ p \Vdash \dot{X} = \dot{Y}$. Note that each $X_\alpha$ has the property that if $p \in X_\alpha$ and $q \geq p$, then $q \in X_\alpha$. Let us call such a sequence $\langle X_\alpha: \alpha \in \omega_1\rangle$ a <u>pseudo-set</u>.

Now suppose $\forall p\ p \Vdash \dot{X} \subseteq \omega_1$ and $|\dot{X}| = \aleph_1$. What does this mean in terms of the pseudo-set representing $\dot{X}$? First note that this proposition can be rewritten as $\forall p\ p \Vdash \dot{X} \subset \omega_1$ and $(\forall\alpha \in \omega_1)(\exists\beta \in \omega_1)\ \alpha < \beta$ and $\alpha \in \dot{X}$. Using the forcing axioms, this is easily translated as $(\forall p \in P)(\forall\alpha \in \omega_1)(\exists q \geq p)(\exists\beta \in \omega_1)\ \alpha < \beta$ and $q \in X_\beta$, a purely combinatorial assertion about $\langle X_\alpha: \alpha \in \omega_1\rangle$.

Finally, suppose $\forall p\ p \Vdash \dot{X}, \dot{Y} \subseteq \omega_1$ and $|\dot{X} \cap \dot{Y}| \leq \aleph_0$. This can be rewritten as

$$\forall p\ p \Vdash \dot{X}, \dot{Y} \subseteq \omega_1 \text{ and } (\exists\alpha \in \omega_1)(\forall\beta \in \omega_1) \text{ if } \beta \in \dot{X} \cap \dot{Y} \text{ then } \beta < \alpha,$$

and it translates as

$$(\forall p \in P)(\exists\alpha \in \omega_1)(\exists q \geq p)(\forall r \geq q)(\forall\beta \in \omega_1) \text{ if } r \in X_\beta \cap Y_\beta \text{ then } \beta < \alpha.$$

Thus a question in $\dot{V}$ about the maximum size of a set of almost disjoint uncountable subsets of $\omega_1$ translates into a question in the real universe of set theory about the maximum size of a collection of pseudo-sets satisfying certain combinatorial conditions. Of course, the answer to the question will depend on the partial ordering P, and one answer is given in the next section. The point, however, is that forcing has been used merely to translate one combinatorial assertion into another. The mathematics of the situation is entirely combinatorial.

## 9. THE NUMBER OF ALMOST DISJOINT SETS.

Here we show that the conclusion of Theorem 2.3 cannot be proved without some special assumptions.

THEOREM 9.1. Suppose $2^{\aleph_1} = \aleph_2$. Let $(P,\leq)$ be any partial ordering with the countable chain condition. Then $(\forall p \in P)\ p \Vdash \varphi$, where $\varphi$ asserts that every family of almost disjoint uncountable subsets of $\omega_1$ has cardinality $\leq \aleph_2$.

PROOF. Our principal tool is the partition relation $\aleph_3 \rightarrow (\aleph_2)^2_{\aleph_1}$, which may be proved from the assumption $2^{\aleph_1} = \aleph_2$ in exactly the same manner as Theorem 3.1.

Suppose $p \Vdash$ "$\dot{F}$ is a one-to-one function with domain $\omega_3$ such that $\{\dot{F}(\alpha): \alpha \in \omega_3\}$ is a family of almost disjoint uncountable subsets of $\omega_1$." We use $\dot{F}_\alpha$ as an abbreviation of $\dot{F}(\alpha)$.

If $\alpha, \beta \in \omega_3$ and $\alpha \neq \beta$, then $p \Vdash |\dot{F}_\alpha \cap \dot{F}_\beta| \leq \aleph_0$, so $p \Vdash (\exists \gamma \in \omega_1) \sup(\dot{F}_\alpha \cap \dot{F}_\beta) = \gamma$. Now let $X_{\alpha\beta} = \{\gamma \in \omega_1: \exists q > p\ q \Vdash \sup(\dot{F}_\alpha \cap \dot{F}_\beta) = \gamma\}$. Note that if $q_i \Vdash \sup(\dot{F}_\alpha \cap \dot{F}_\beta) = \gamma_i$ for $i = 1, 2$, and $\gamma_1 \neq \gamma_2$, then $q_1$ and $q_2$ are incompatible. Since $P$ has the countable chain condition, then, $X_{\alpha\beta}$ must be countable. Let $\gamma_{\alpha\beta}$ be the supremum of the ordinals in $X_{\alpha\beta}$. Then $\gamma_{\alpha\beta} \in \omega_1$ and clearly $p \Vdash \dot{F}_\alpha \cap \dot{F}_\beta \subseteq \gamma_{\alpha\beta}$.

Now define a partition function $f: [\omega_3]^2 \rightarrow \omega_1$ by $f\{\alpha,\beta\} = \gamma_{\alpha\beta}$. By $\aleph_3 \rightarrow (\aleph_2)^2_{\aleph_1}$, there is a homogeneous set $B \subseteq \omega_3$ such that $|B| = \aleph_2$. Say $f\{\alpha,\beta\} = \gamma$ for all $\{\alpha,\beta\} \in [B]^2$. Let $\dot{H}_\alpha$ be a symbol to denote $\dot{F}_\alpha - \{\delta: \delta < \gamma\}$. Note that if $\{\alpha,\beta\} \in [B]^2$, then $p \Vdash \dot{H}_\alpha \cap \dot{H}_\beta = 0$ and $\dot{H}_\alpha$ and $\dot{H}_\beta$ are uncountable. But now $p \Vdash$ "$\{\dot{H}_\alpha: \alpha \in B\}$ is a set of pairwise disjoint nonempty subsets of $\omega_1$", and this is a contradiction since $p \Vdash |B| = \aleph_2$ (by Theorem 6.1). This completes the proof.

Two remarks may help explain the significance of Theorem 9.1. First, it is easy to find partial orderings $(P,\leq)$ with the c.c.c. such that $\forall p \in P\ p \Vdash 2^{\aleph_1} \geq \aleph_3$. For example, consider the partial ordering $P$ of §5, defined for a set $A_0$ such that $|A_0| = \aleph_3$. For this $P$ one can either prove directly that $\forall p\ p \Vdash 2^{\aleph_0} \geq \aleph_3$ (so $2^{\aleph_1} \geq \aleph_3$ also) or observe that $\forall p\ p \Vdash \aleph_3 \not\rightarrow [\aleph_1]^2_{\aleph_0}$ and that Theorem 3.1 can be extended to read that if $2^{\aleph_0} = \aleph_2$ then $\aleph_3 \rightarrow (\aleph_1)^2_{\aleph_0}$.

Thus it follows, as in the discussion in § 4, that if $ZFC + 2^{\aleph_1} = \aleph_2$ is consistent then so is $ZFC + 2^{\aleph_1} \geq \aleph_3 + \varphi$, and this shows that the conclusion of Theorem 2.3 is not provable in ZFC alone.

Second, there is a famous theorem of Gödel (see any advanced set theory text) which says that if ZF is consistent (ZF is ZFC without the axiom of choice) then so is $ZFC + (\forall\alpha) 2^{\aleph_\alpha} = \aleph_{\alpha+1}$. Putting these two remarks together, we obtain:

COROLLARY 9.2. If ZF is consistent, then so is $ZFC + 2^{\aleph_1} \geq \aleph_3 + \varphi$.

Once again the proof of Theorem 9.1 was almost entirely combinatorial. Note also a very convenient feature of the proof: even though $2^{\aleph_1} = \aleph_2$ is false in $\dot{V}$ we are allowed to assume it (in the real universe) for the purposes of the proof.

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