LECTURE 7

→ $\boxed{\text{Ajtai's thm}}$, part 2

RECALL:

- $0, 1, \neg, \vee$ (binary)
- F_d ... depth $\leq d$ subsystem
- $\neg \text{onto}PHP_n$: UNSAT SET OF CLAUSES in ATOMS p_{ij}

  $i \in [n+1], j \in [n]$

AIM: AJTAI'S THM (W/ IMPROVEMENTS)

$\forall d \geq 2\ \exists \delta > 0\ \forall n >> 0$

IN ANY F_d-REFUT. OF $\neg$ onto PHP_n

MUST OCCUR $\geq 2^{n^\delta}$ DIFFERENT

FLAS AS SUBFLAS.

$\Rightarrow$ SIZE $\geq 2^{n^\delta}$

LAST TIME:

k-eval. of Γ ← CLOSED UNDER SUBFLAS

$(H, S) : \varphi \in \Gamma \longrightarrow H_\varphi \subseteq S_\varphi$: k-tree

φ is (H,S)-true (OR "TRUE" W.R.T. (H,S))

$\overset{\text{df.}}{\Longleftrightarrow}$ $H_\varphi = S_\varphi$

LEMMA 6

ALL CLAUSES of $\neg$ ontoPHP_n are (H,S)-TRUE IF $k \leq n-2$.

LEMMA 7

IF ALL HYPOTHESES OF AN F-RULE ARE (H,S)-TRUE, SO IS THE CONCLUSION, ASSUMING $k \leq n/c_F$

CONST. DEPENDING ON F ONLY

TODAY:

$0 < \delta < 5^{-d}$

"LEMMA": IF $|\Gamma| \le 2^{n^{\delta}}$ (SMALL) THEN A k-EVAL. (H,S) EXISTS WITH $k \le n^{\delta}$.

$dp(\varphi) \le d$ all $\varphi \in \Gamma$

Γ closed under subflas

PRF OF THE THM: $\Gamma :=$ ALL FLAS OCCURING IN A REFUT.

TAKE (H,S) FROM, APPLY LG & LT

TO CONCLUDE THAT 0 IS (H,S)-TRUE.

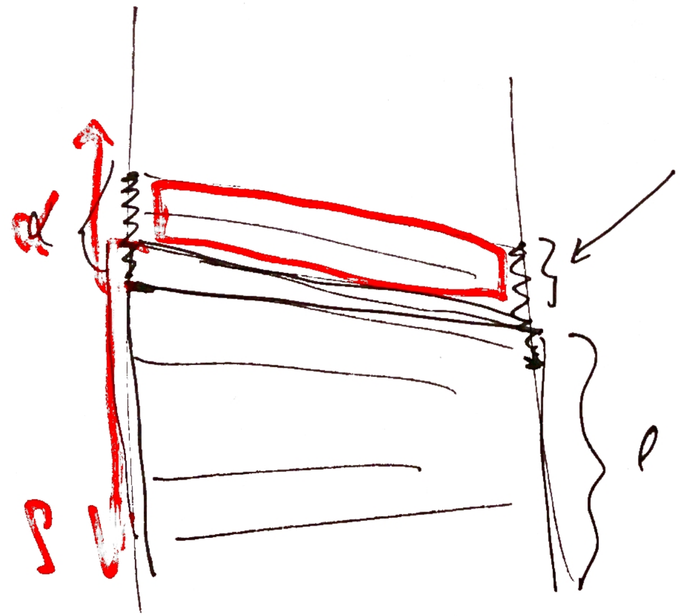
THE LAST FLA

$\lightning$

6.

$\alpha, \rho \in$ Raps (call greek Letters $\in$ Raps)

$$\alpha^{\rho} :\underset{\text{df.}}{=} \begin{cases} \alpha \smile \rho, & \text{if } \alpha \parallel \rho \\ \text{UNDEFINED}, & \text{if } \alpha \nparallel \rho \end{cases}$$



$\alpha^{\rho} = \alpha \smile \rho$

$H^\rho := \{\alpha^\rho \mid \alpha \in H\}$

$D^\rho := D \setminus dom(\rho)$

$R^\rho := R \setminus rng(\rho)$

$n_\rho := |R^\rho| \; (= n - |\rho|)$

$$p_{ij}^\rho := \begin{cases} 1, & (i,j) \in \rho \\ 0, & (i,j) \perp \rho \\ p_{ij}, & \text{otherwise} \end{cases}$$

$$(\neg \mathrm{onto}\,PHP_n)^\rho \simeq \neg \mathrm{onto}\,PHP_{n_\rho}$$

L15.2.1 $\forall \rho \in \text{Maps}$

(i) $H \lhd S \implies H^{\rho} \lhd S^{\rho}$

(ii) $|\rho| + h(S) \leq n \implies S^{\rho}$ is a tree over D^{ρ}, R'

(iii) $H \lhd S \implies S^{\rho}(H^{\rho}) = (S(H))^{\rho}$.

PRF's: DIRECTLY FROM DEFINITIONS. $\square$

[Sec. 15.2]

<u>COROLLARY</u>: $|\rho| + k \leq n$ and (H, S) is a k-eval. OF Δ

$$\Downarrow$$

(H^ρ, S^ρ) is A k-EVAL. OF Δ^ρ.

<u>PRF</u>: EX: A COND. FOR $\varphi = \bigvee_j \psi_j$ is

$\bigcup_j H_{\psi_j} \lhd S_\varphi$ & $H_\varphi = S_\varphi\left(\bigcup_j H_{\psi_j}\right)$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$\bigcup_j H_{\psi_j}^\rho \lhd S_\varphi^\rho$ $\qquad H_\varphi^\rho = (\ldots\ldots)^\rho = S_\varphi^\rho\left(\bigcup_j H_{\psi_j}^\rho\right)$. $\square$

THE IDEA OF A CONSTRUCTION:

$\Gamma_i := \{\varphi \in \Gamma \mid dp(\varphi) \leq i\}$, so $\Gamma_d = \Gamma$

(H_0, S_0) for Γ_0 ... CANONICAL (JUST ATOMS = CNT's)

(H_i, S_i) FOR Γ_i

$\downarrow$

(H_{i+1}, S_{i+1}) FOR Γ_{i+1}

→ CANONICAL FOR $\neg\varphi \in \Gamma_{i+1}$, $\varphi \in \Gamma_i$

DISJUNCTIONS $\varphi \in \Gamma_{i+1}$, $\varphi = \bigvee_j \varphi_j$, $\varphi_j \in \Gamma_i$

$\downarrow$

NEED $S_\varphi \supseteq \bigcup_j H_{\varphi_j}$

determined by (H_i, S_i)

$\Downarrow$

11.

SO THE <u>TASK FACING US IS</u>:

given H (OR MANY H_i, ONE FOR EACH DISJUNCTION $\in \Gamma_{i+1} - \Gamma_i$)

find k-tree $S \rhd H$.

<u>WILL DO</u>: WE FIND SUCH S AFTER A RESTRICTION BY SOME ρ,

i.E. $S \rhd \boxed{H^{\rho}}$

L 15.2.2 [TECHNICAL HIGH POINT]

LET $0 < \delta < \varepsilon < 1/5$, $H_i \subseteq \mathrm{Maps}$ for $i \le s$.

ASSUME $h(H_i) = \max\{|\alpha| \mid \alpha \in H_i\} \le k$, all $i \le s$,

AND $n >> 1$, $k \le n^{\delta}$, $s \le 2^{n^{\delta}}$.

THEN $\exists \rho \in \mathrm{Maps}$, $n_\rho = n - |\rho| = n^{\varepsilon}$

AND

$\exists S_i\ (i \le s)$, S_i a k-TREE OVER D^{ρ}, R^{ρ}

(i) $S_i \rhd H_i{}^{\rho}$, all $i \le s$.

(ii) $h(S_i) \le k$, all $i \le s$.

$\square$

FIRST

$\boxed{L\ 15.2.3}$:

LET $0 < \delta < \varepsilon < 5^{-d}$. THEN FOR ALL $n \gg 1$,

ALL Γ : CLOSED UNDER SUBFLAS

$dp(\Gamma) \leq d$

$|\Gamma| \leq 2^{n^\delta}$

EXIST ρ, $|\rho| = n - n^\varepsilon$ AND

$\exists$ k-eval. of Γ^ρ, with $k \leq n^\delta$.

PROOF FROM L 15.2.2 :

- AS OUTLINED ON SLIDE 11

- START WITH P_0 - CANONICAL

~~GIVEN FOR i≤ WE HAVE:~~

PICK $\varepsilon_0 < 1/15 : \varepsilon = \varepsilon_0^{d}$.

PUT : $\rho_0 := \emptyset$, (H_0, S_0) - THE CANONICAL

$i \to i+1$: WE HAVE $\rho_0 \subseteq \rho_1 \subseteq \dots \subseteq \rho_i$

• $n_{\rho_i} = n - |\rho_i| = n^{\varepsilon_0^{i}}$

AND (H_i, S_i): n^{δ}-EVAL OF $\Gamma_i^{\rho_i}$

APPLY L 15.2.3 : .... ~~EX~~ RESTR. ρ ON D^{ρ_i}, R^{ρ_i}

IS THE SAME THING AS RESTR. $\rho \supseteq \rho_i$.

GIVES US $\rho : \rho_{i+1}$

$\hookrightarrow (H_{i+1}, S_{i+1})$: n^{δ}-EVAL OF $\Gamma_{i+1}^{\rho_{i+1}}$

(NOTE : $(\Gamma_i^{\rho_i})^{\rho} = \Gamma_i^{\rho_{i+1}}$)

- BY L 15.2.1: ALL $(H_i, S_i)_{i \le d}$ ARE n^δ-EVAL'S OF $\Gamma_i^{\rho_i}$.

$\square_{L\,15.2.3}$

THIS YIELDS A PROOF OF AJTAI'S THM

↳ SEE SLIDE 6

AFTER ρ: $(\neg\,\mathrm{onto}\,PHP_n)^\rho = \neg\,\mathrm{onto}\,PHP_{n^\rho}$

SO (H_d, S_d) n^δ-EVAL. OF Γ^{ρ_d}

GIVES (H, S) n^δ-EVAL OF ~~ALL~~ FLAS IN A ~~PRO~~ REFUT. OF

AS $k = n^\delta << n^\varepsilon = |n_\rho|$, L 6.7 APPLY.

$\square_{THM.}$

THE REST OF THE LECTURE:

A PROOF OF L 15.2.2

• WE HAVE: $0 < \delta < \varepsilon < 1/5$

$k \le n^{\delta}$, $s \le 2^{n^{\delta}}$

$\{H_i\}_{i \le s}$, $h(H_i) \le k$

• WE WANT: ρ, $|\rho| = n - n^{\varepsilon}$

$\forall i \le s$ $\exists$ k-tree S_i over D^{ρ}, R^{ρ}

$S_i \rhd H_i^{\rho}$.

LET US START WITH <u>ONE</u> H

(GENERALIZATION TO H_i's WILL BE EASY)

<u>enumerate H</u> : $h^1, h^2, \ldots$

[USING LETTERS h TO BE IN LINE W THE BOOK]

<u>2-PLAYER GAME</u>

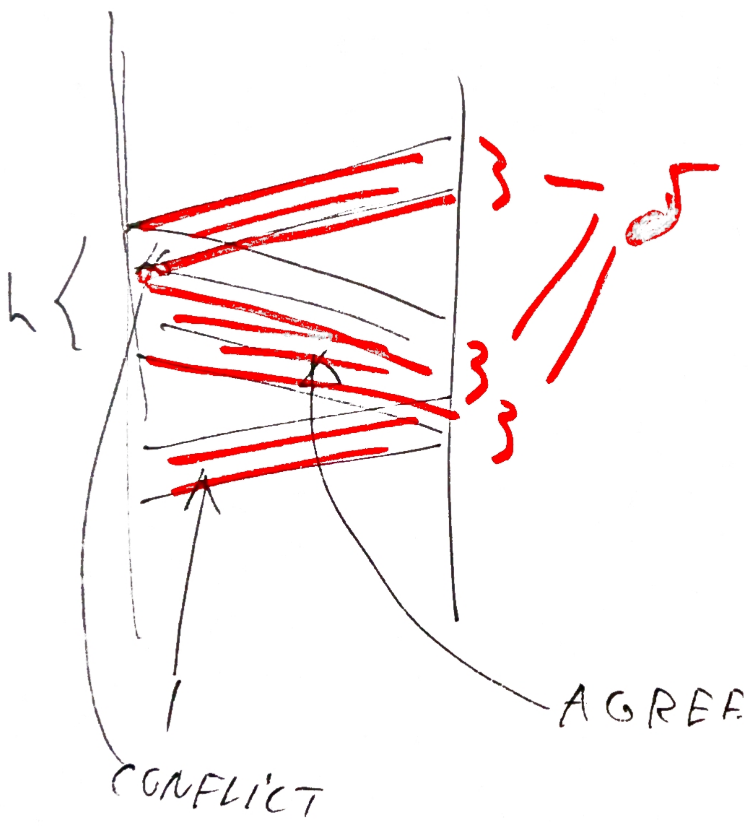
- I : WILL BE CHOOSING $h \in H$
- II : WILL REPLY WITH ε drops.

K.

STEP 1 :

I : picks $h_1 :=$ the FIRST h ~~s.t.~~

II : picks δ_1 : $\leq$-minimal s.t. $dom(h_1) = dom(\delta_1)$
$rng(h_1) = rnd(\delta_1)$



- $\delta_1 = h_1$ as $\delta_2 \supseteq h$ as $h \in H$
  $\Rightarrow$ GAME STOPS

- $\delta_1 \perp h_1$ $\Rightarrow$ NEXT ROUND.

BEFORE ROUND $t \geq 2$

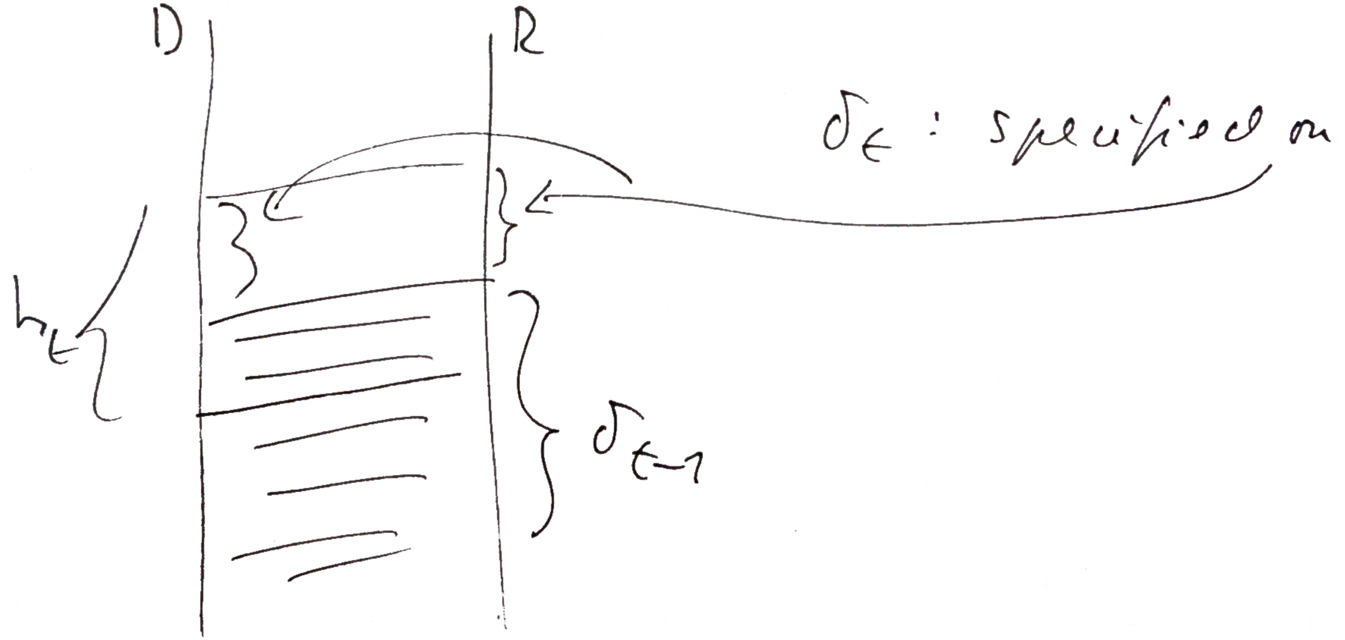
I: $h_1, \dots, h_{t-1}$

II: $\delta_1 \subseteq \delta_2 \subseteq \dots \subseteq \delta_{t-1}$

ROUND t: I: picks h_t := THE FIRST $h \in H$: $h \parallel \delta_{t-1}$

II: picks $\delta_t \supseteq \delta_{t-1}$: $\subseteq$-min s.t. $\mathrm{dom}(h_t) \subseteq \mathrm{dom}(\delta_t)$

$\mathrm{rng}(h_t) \subseteq \mathrm{rng}(\delta_t)$



RULES : $\delta_t \supseteq h$, some $h \in H$ $\Rightarrow$ $\boxed{\text{GAME STOPS}}$

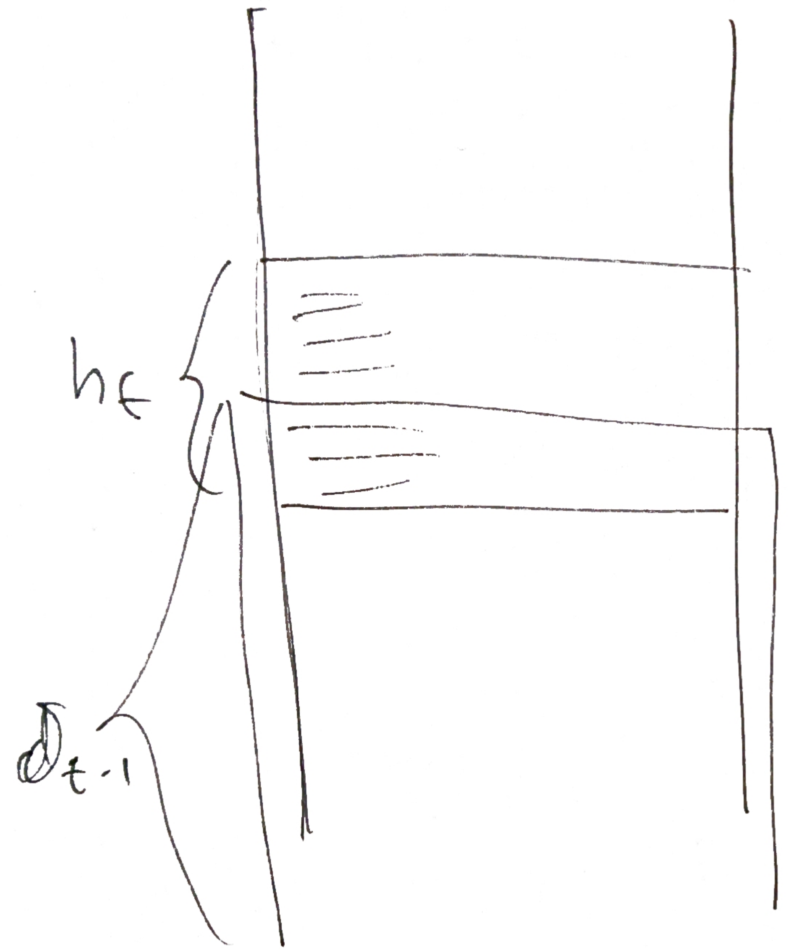
$\delta_t \perp h$, all $h \in H$ $\Rightarrow$

(i.e. all H is EXHAUSTED)

CLAIM 1 : $S = \{\delta_t \mid \delta_1 \subseteq \ldots \subseteq \delta_t$ is A FINISHED PLAY$\}$ is a TREE AND $S \rhd H$.

$$h(S) = \max\{|\delta_t| \,/\, \delta_t \in S\}.$$

$\square$



CRITICAL PAIRS

δ_t NEEDS TO BE SPECIFIED HERE.

<u>EX.</u> $H = \{(i,j) \mid \text{all } i,j\}$

$\Rightarrow$ ANY ~~GAxx~~ PLAY MUST EXHAUT ALL j

$\Rightarrow$ $\geq n/2$ CRITICAL PAIRS.

<u>CLAIM</u>: $\exists f \in$ maps, $n_f = n^{\varepsilon}$ s.t.

$\forall$ play on H^f has $\leq k/2$ CRITICAL PAIRS.

$\Downarrow$

$\delta_t \in \mathcal{S} \Rightarrow |\delta_t| \leq 2 - \ell/2 = 4$, i.e.

S is A <u>4</u>-TREE.

PROOF OF CLAIM 2 – BY CONTRADICTION

ASSUME :

$$\forall \rho \ldots \exists \text{ plog } \delta_1 \le \ldots \le \delta_t \text{ with } > k/2 \text{ CRIT. PAIRS}$$

$$\Downarrow$$

TRUNCATE δ_t TO $k/2$ PAIRS IN TOTAL

WE ARE GOING TO USE THIS TO SPECIFY ANY ρ BY LESS BITS THAN IT IS POSSIBLE BY COUNTING

$$\vdots$$

$$\# \text{ of } \rho\text{'s} > \# \text{ of DATA SETS}$$

PUT

$\tau := \rho \cup$ all critical pairs

$(\tau \in \mathrm{Maps})$

$|\tau| = |\rho| + \varepsilon/2 \geq n - n^{\varepsilon}$

FROM τ WE CANNOT RECONSTRUCT ρ

BUT WE CAN FIND h_1: the FIRST $h \in H_1$ $h \,||\, \tau$

- $\delta_1 \ldots$ def. on k_1 crit. pairs

$$\#\text{ OF OPTIONS} \leq \binom{k}{k_1} \cdot \Big[\underbrace{n^{\varepsilon}}_{\text{values}}\underbrace{(n^{\varepsilon}+1)}_{\text{preimages}}\Big]^{k_1} \leq \left(k^{1/2} \cdot n^{\varepsilon}\right)^{2k_1}$$

NOW TAKE

$\tau' :=$ REPLACE IN τ ALL CRITICAL PAIRS IN h_1

BY σ_1

$\Downarrow$

τ' DETERMINES $h_2 :=$ the FIRST $h \in H$, $h \| \tau'$

$\Downarrow$

again $\leq (k^{1/2} \cdot n^{\varepsilon})^{2k_2}$ options for σ_2

($k_2 :=$ # of CRIT. PAIRS in h_2)

ALL TOGETHER:

$$\underbrace{(\#\text{ of } \Sigma\text{'s})}_{b} \cdot \prod_{i \leq \ell} \left(k^{1/2} \cdot n^{\varepsilon}\right)^{2k_i} = a \left(k^{1/2} \cdot n^{\varepsilon}\right)^{k}$$

THIS BOUND THE NB. OF DATA SETS SPECIFYING ANY ρ

$$b := \binom{n+1}{n^{\varepsilon} - k/2 + 1} \binom{n}{n^{\varepsilon} - \ell/2} \left(n - n^{\varepsilon} - \ell/2\right)!$$

$$a := \#\text{ of } \rho\text{'s} = \binom{n+1}{n^{\varepsilon} + 1} \binom{n}{n^{\varepsilon}} \left(n - n^{\varepsilon}\right)!$$

TO REACH A CONTRADICTION FOR ONE H

IT SUFFICES TO VERIFY:

$$a > b \cdot (k^{1/2} \cdot n^{\varepsilon})^{4}$$

$$\text{''} \qquad \text{FOR } \{H_i\}_{i \le s}$$

IT SUFFICES IF:

$$\boxed{a > s \cdot b \cdot (k^{1/2} \cdot n^{\varepsilon})^{4}}$$

~~EASY~~ ELEMENTARY COMPUT.

FOR THE PARAMETERS, USING $0 < \delta < \varepsilon < 1/5$ AND $n >> 1$. $\square$

L15.2.2