- $\mathcal{L} \supseteq \mathcal{L}_{PA}$, finite
- $T \supseteq Q$ (Robinson's Q), $\boxed{\text{recursive}}$ ← will change.
- $\mathbb{N} \models T$ (can be removed)
- T consistent

$$T \vdash \varphi \iff \mathbb{N} \models \underbrace{Pr_T}_{\Sigma_1}(\ulcorner\varphi\urcorner)$$

Löb's conditions:

(1) $T \vdash \varphi \Rightarrow T \vdash Pr_T(\ulcorner\varphi\urcorner)$

(2) $T \vdash Pr_T(\ulcorner\varphi\urcorner) \to Pr_T(\ulcorner Pr_T(\ulcorner\varphi\urcorner)\urcorner)$

(3) $T \vdash Pr(\ulcorner\alpha\urcorner) \wedge Pr(\ulcorner\alpha \to \beta\urcorner) \to Pr(\ulcorner\beta\urcorner)$

Gödel's diagonal lemma:

$$T \vdash \sigma \equiv \neg Pr_T(\ulcorner\sigma\urcorner)$$

some σ.

Gödel's First Th: $T \nvdash \sigma$ but $\mathbb{N} \models \sigma$.

Hence $T \nvdash \neg\sigma$ too.

Prf: Assume

$$\begin{tikzcd} T\vdash\sigma \arrow[r, Rightarrow, "(1)"'] \arrow[d, Rightarrow, "(4)"] & T\vdash \Pr(\ulcorner\sigma\urcorner) \arrow[d, phantom, "\Big\}"] \\ T\vdash\neg\Pr(\ulcorner\sigma\urcorner) & \Rightarrow T \text{ inconsistent } \lightning \end{tikzcd}$$

□

Gödel's Second Theorem

For $\mathrm{Con}_T := \neg\Pr(\ulcorner 0\neq 0\urcorner)$, $T\vdash \mathrm{Con}_T \to \sigma$, where σ is from (4). Hence $T\nvdash \mathrm{Con}_T$.

Prf: Argue inside T:

$$\begin{tikzcd} \text{Assume } \neg\sigma. \arrow[r, Rightarrow, "\substack{(4)\\ \text{modus}\\ \text{ponens}}"'] & \Pr(\ulcorner\sigma\urcorner) \arrow[r, Rightarrow, "(2)"'] \arrow[d, Rightarrow, "\substack{(4),\,MP\\(1)}"] & \Pr(\ulcorner\Pr(\ulcorner\sigma\urcorner)\urcorner) \\ & \Pr(\ulcorner\neg\Pr(\ulcorner\sigma\urcorner)\urcorner) \arrow[r, Rightarrow, "(3)"] & \Pr(\ulcorner\overset{0\neq 0}{\bot}\urcorner). \end{tikzcd}$$

So we proved:

$$T\vdash\neg\sigma \Rightarrow \neg\mathrm{Con}_T .$$

As $T\nvdash\sigma$, $T\nvdash \mathrm{Con}_T$ follows.

□

Further assumptions on T/L (plus notation)

- **Numerals** $\underline{n}$: $\underline{0} := 0$, $\underline{1} := 1$, $\underline{2n} := (1+1)\cdot\underline{n}$,

  $\underline{2n+1} := (\underline{2n} + 1)$

  Note: $|\underline{n}| = O(\log n)$.

- There is efficient coding of sequences $\omega \in \mathbb{N}^*$ by numbers, s.t. for

  $\ulcorner \omega \urcorner :=$ "the numeral of the nb. coding ω"

  it holds:

  $$|\ulcorner \omega \urcorner| = O(\sum_{i<k} |a_i|) \quad , \text{if } \omega = (a_0, \ldots, a_{k-1})$$

  In particular, if $\omega \in \{0,1\}^*$: $|\ulcorner \omega \urcorner| = O(|\omega|)$.

- **Provability predicate** :

  $$Prf(x, y, z)$$

  ~~S.t.~~ formalizing : " x is a (finite) list of L-axioms, z is an L-fla and y is a proof of z from x " , such that

  (i) $\mathbb{N} \models Prf(\ulcorner S \urcorner, \ulcorner \pi \urcorner, \ulcorner \varphi \urcorner)$ iff

  " π is an S-proof of φ "

(ii) For all $a, b, c \in \mathbb{N}$, if $\mathbb{N} \models Prf(a,b,c)$ then $Prf(\ulcorner \bar{a} \urcorner, \ulcorner \bar{b} \urcorner, \ulcorner \bar{c} \urcorner)$ has a T-proof of size (= nb. of symbols) $(|a|+|b|+|c|)^{O(1)}$.

- T is finite (this simplifies technicalities but T having a p-time set of ax's is o.k.).

- $Pr_T(z) :\equiv \exists y \, Prf(\ulcorner T \urcorner, y, z)$

- $Pr_T^w(z) :\equiv \exists y \, (|y| \leq w) \, Prf(\ulcorner T \urcorner, y, z)$.

We shall denote this also $T \vdash_w z$.

Löb's conditions modified for Pr_T^+

(1) $T \vdash_m \varphi \Rightarrow T \vdash_{mc} Pr^m(\ulcorner \varphi \urcorner)$

(2) $T \vdash Pr_T^{+}(z) \to Pr_T^{+c}(\ulcorner Pr^{+}(z) \urcorner)$

(3) $T \vdash (Pr_T^{x}(z_1) \wedge Pr_T^{+}(z_1 \to z_2)) \to Pr_T^{10+}(z_2)$

Better Gödel's diagonalisation:

(4) $T \vdash \sigma(x) \equiv (\neg Pr^{+}(\ulcorner \sigma(\dot{x}) \urcorner))$

This is an instance of general diag.-lemma saying, that for any $\varphi(x)$ there is $\psi(x)$ s.t.

$$T \vdash \psi(x) \equiv \varphi(\ulcorner \psi(\dot{x}) \urcorner).$$

Revised First thm : For all $n \geq 1$,

$\mathbb{N} \models \delta(\underline{n})$ but $T \not\vdash_n \delta(\underline{n})$.

Prf : (analogous as before)

$$\begin{tikzcd} T \vdash_n \delta(\underline{n}) \arrow[r, Rightarrow, "(1)"'] \arrow[d, Rightarrow, "(4)"] & T \vdash Pr^n(\ulcorner \delta(\underline{n}) \urcorner) \\ T \vdash \neg Pr^n(\ulcorner \delta(\underline{n}) \urcorner) & \end{tikzcd}$$

$\Big\}$ ↯ with the consistency of T.

□

Second Thm. revised [H. Friedman '79, P. Pudlák '86]
(= Quantitative Gödel's thm)

For $Con_T(x) := \neg Pr_T^x(\ulcorner 0 \neq 0 \urcorner)$

There is $\varepsilon > 0$ s.t. for all $n \geq 1$:

$$T \not\vdash_{n^\varepsilon} Con_T(\underline{n}).$$

Note: $|Con_T(\underline{n})| = O(\log n)$, so the lower bound is exponential.

$\longrightarrow$ -

Lemma: If $T \vdash \varphi(x)$ then for all $n \geq 1$,

$$T \vdash_{O(\log n)} \varphi(\underline{n}).$$

Pf: Use substitution and $|\underline{n}| = O(\log n)$. $\square$

Cor. $T \not\vdash Con_T$ $\square$.

Prf:

(1) Combining (4) and (2), T proves

$$\neg\sigma(x) \longrightarrow \Pr^{x^c}(\ulcorner \Pr^{\underline{x}}(\ulcorner\sigma(\underline{x})\urcorner)\urcorner)$$

(2) Using (4) and formalized Lemma in T, T proves:

$$\Pr^{O(\log x)}(\ulcorner\neg\sigma(\underline{x})\urcorner \longrightarrow \neg\Pr^{\underline{x}}(\ulcorner\sigma(\underline{x})\urcorner)\urcorner)$$

So with (3), (1): it also proves:

$$\neg\sigma(x) \longrightarrow \Pr^{O(\log x + x)}(\ulcorner\neg\Pr^{\underline{x}}(\ulcorner\sigma(\underline{x})\urcorner))$$

(3) (1) & (2) imply that for some d, T proves:

$$\neg\sigma(x) \longrightarrow \Pr^{x^d}(\ulcorner 0\neq 0\urcorner)$$

(we just need $x^d > O(\log x + x + \xi x + {}^{c})$).

(4) (3) + Lemma imply for all $m \geq 1$:

$$T \vdash_{O(\log m)} \neg\sigma(\underline{m}) \longrightarrow \cancel{\Pr^{\underline{m}^d}(\ulcorner 0 =} \neg\mathrm{Con}_T(\underline{m}^d)$$

(5) Choose $\varepsilon > 0$ so small that $(m^d)^{\varepsilon} < m/2$. As by the First Thm: $T \nvdash_m \sigma(\underline{m})$, we cannot have

~~$T \vdash_{(m^d)^{\varepsilon}/2} \mathrm{Con}_T(\underline{m}^d)$~~ $T \vdash_{m/2} \mathrm{Con}_T(\underline{m}^d)$ as that would yield - with (4) - that $T \vdash_m \sigma(\underline{m})$. □

Remarks

What if we want $S \vdash_{?} \mathrm{Con}_T(\underline{n})$.

(a) S >> T, say $S \vdash \mathrm{Con}_T$. Then, by Lemma,

$$S \vdash_{O(\log n)} \mathrm{Con}_T(\underline{n}).$$

(b) S << T, even S fixed : OPEN.

Conj. : There is no fixed (finite, ...) S s.t. for all T (finite, ...) $\exists c \geq 1$ t/u :

$$S \vdash_{n^c} \mathrm{Con}_T(\underline{n}).$$

(c) Upper bound (Pudlák '87) For S = T there are poly upper bounds:

$$T \vdash_{n^{O(1)}} \mathrm{Con}_T(\underline{n}).$$

[Remark: this is exponentially better than exhaustive search!]