Gödel's First Incompleteness Thm.

We have established 3 statements:

(1) If T is recursive ($\Leftrightarrow$ it is alg.-decidable if φ is a T-axiom), consistent and complete then it is alg. decidable whether $T \underset{?}{\vdash} \varphi$.

[Lect.8, the alg searches through all strings looking for a T-proof of φ (outputs YES) or of $\neg\varphi$ (outputs NO).]

(2) Σ_1-completeness of Q. } separate notes

(3) Σ_1-definability of RE sets. }

Gödel's theorem (1931)

Let T be a theory s.t.

(i) $L_T \supseteq L_Q$ and $T \supseteq Q$.

(ii) T is recursive.

(iii) For all L_Q Σ_1-sentences σ, $T \vdash \sigma \Rightarrow \mathbb{N} \models \sigma$.
(This implies consistency of T.)

Then T is incomplete. In particular, there are Σ_1-sentences α s.t. $\mathbb{N} \models \neg\alpha$ but $T \nvdash \neg\alpha$.

Prf: Take $H \in RE \setminus R$ (e.g. HALT).

By (3) there is Σ_1-fla $\varphi(x)$ s.t. for all $n \geq 0$:

$$n \in H \iff \mathbb{N} \models \varphi(n)$$

But by (2) this is $\iff \mathbb{Q} \vdash \varphi(s_n)$.

As $H \notin R$, we cannot algorithmically decide all instances

$$\mathbb{Q} \underset{?}{\vdash} \varphi(s_n).$$

By (1) this implies that T is incomplete.

□ thm.

Remarks:

1) (iii) follows from $\mathbb{N} \models T$ (the $L_{\mathbb{Q}}$-part).

2) (iii) can be weakened - by a different proof - to "consistency of T".

3) (i) can be weakened to "T interprets $L_{\mathbb{Q}}$ and $\mathbb{Q}$". For example, ZFC defines $\mathbb{N}$, interprets on it $L_{\mathbb{Q}}$ (even if $L_{\mathbb{Q}} \not\subseteq L_{ZFC}$) and proves $\mathbb{N} \models \mathbb{Q}$. Thus the thm applies to ZFC too.

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