# § Peano & Robinson Arithmetics

**language** $L_{PA} = L_Q$ : $0$

$s(x)$, [successor function]

$+, \cdot$

$\leq$

**Robinson's Q:** axioms are universal closures of

Q1 $s(x) \neq 0$

Q2 $s(x) = s(y) \to x = y$

Q3 $x \neq 0 \to \exists y \, (s(y) = x)$

Q4 $x + 0 = x$

Q5 $x + s(y) = s(x+y)$

Q6 $x \cdot 0 = 0$

Q7 $x \cdot s(y) = (x \cdot y) + x$

Q8 $x \leq y \equiv (\exists z, z + x = y)$.

**Fact 1:** $\mathbb{N} \models Q$, where $0, +, \cdot, \leq$ are interpreted as usual and $s(x) := x + 1$

$\square$

Fact 2 Q proves the following three formula we shall use later:

Q9 : $0 \leq x$ [by Q4 + Q8 $\exists$, choose $z := x$]

Q10 : $S(x) \neq 0$ [by Q1, Q5 and Q8 :

$z + S(x) = 0 \xrightarrow{Q5} S(z + x) = 0$ $\lightning$ with Q1]

Q11 : $x < y \iff S(x) \leq S(y)$

$\Rightarrow$: use Q5 + Q8 : $z + x = y \Rightarrow z + S(x) \underset{Q5}{=} S(z + x) = S(y)$

$\Leftarrow$ : use Q2, Q5 + Q8 : $z + S(x) = S(y) \underset{Q5}{\Rightarrow}$
$S(z + x) = S(y) \underset{Q2}{\Rightarrow} z + x = y$.

$\square$

Peano arithmetic PA : axioms are Q plus the scheme of induction: for any f/o $\varphi(x, \bar{z})$ the universal closure of $Ind_\varphi$ is an axiom of PA. $Ind_\varphi$:

$$\varphi(0, \bar{z}) \wedge (\forall x, \varphi(x, \bar{z}) \rightarrow \varphi(S(x), \bar{z})) \longrightarrow \forall x\, \varphi(x, \bar{z})$$

<u>Fact 3</u> : PA is mutually interpretable with "theory of finite sets" : $ZF_{fin} := ZF$ with the ax. of $\infty$ replaced by its negation.

<u>Idea:</u> $PA \leq ZF_{fin}$ : use usual def. of ordinals, $\neg Ax\infty$ then implies IND

$ZF_{fin} \leq PA$ : to each $n$ attach a hereditarily finite set $n^*$ by inductive process:

$0^* := \{\}$

$n^* := \{k_1^*, \ldots, k_n^*\}$, where

$n = 2^{k_1} + \ldots + 2^{k_n}$, $k_1 > \ldots > k_n$.

$\square$