Proof Theory

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Herbrand's thm.

Reduction of first-order logic to propositional logic

<u>Thm</u>: Let T be a theory axiomatized by purely universal formulas.
Suppose $T \models \forall \bar{x} \, \exists y_1, \dots, y_k \, B(\bar{x}, \bar{y})$,
with $B(\bar{x}, \bar{y})$-quantifier-free.
Then, there is a finite sequence of terms $t_{i,j} = t_{i,j}(\bar{x})$, so that:

$$T \vdash \forall \bar{x} \underbrace{\bigvee_{i=1}^{r} B(\bar{x}, t_{i,1}, \dots, t_{i,k})}_{\text{Herbrand's disjunction}}.$$

"Proving witnessing theorem provides you explicit witnesses".

$\perp$.

Pr: (sketch)

Assume T is axiomatized by quantifier-free formulas (possibly with parameters). Let $\mathcal{T}$ be the set of sequents $\longrightarrow A$, with $A \in T$.

Since $T \models \forall \bar{x}\, \exists \bar{y}\, B(\bar{x}, \bar{y})$, it follows that there is a $LK_{\mathcal{T}}$-proof of the sequent $\longrightarrow \exists \bar{y}\, B(\bar{a}, \bar{y})$.

By free-cut elimination, there is a free-cut free proof P of the above sequent.

Note that all the $\mathcal{T}$-sequents are quantifier-free, thus all the cut formulas in P a quantifier-free, since they are anchored by $\mathcal{T}$.

Herbrand's thm. end

By a subformula property,
all the non-quantifier-free f-las in P
are subformulas (generalized) of $\exists \bar{y} B(\bar{a}, \bar{y})$.

Thus, such f-las are necessary
of the form $\exists y_{j}, \ldots, y_{k} B(\bar{a}, t_{1}, \ldots, t_{j-1}, y_{j}, \ldots, y_{k})$.

Note that such f-las necessary
appear on the right side of the
sequent they belong to.

It can then be shown that P
can be modified to a valid $LK_{\mathcal{T}}$-proof
of a sequent of the form

$$\longrightarrow B(\bar{a}, t_{1,1}, \ldots, t_{1,k}), \ldots\ldots$$
$$B(\bar{a}, t_{r,1}, \ldots, t_{r,k}).$$

The idea is to remove all $\exists$:right
inferences in P and remove all existential
quantifiers ■

General Herbrand's thm.

There are alternadive (model-theoretic) proofs of the previous thm.

Note that if a quantifies-free f-la C is provable in an universal theory T, then C is a consequence of a finite set of substitution instances of axioms of T.

In particular, if T is the null theory, then C is a consequence of instances of equality axioms (such C is called <u>quasi tautology</u>)

If C does not involve equality symbol, then C is actually a tautology.

Thus, Herbrand's thm. reduces provability in first-order, to generation of (quasi) tautologies.

General Herbrand's thm.
cond.

There are two ways one can generalize the Herbrand's thm. to cases besides Π_2-thms in purely universal theories.

<u>First</u>:
Enlarge the original language by adding new function symbols, both for all the existential quantifiers appearing in axioms of T (<u>skolemization</u>) and for universal quantifiers appearing in a formula one is proving (<u>herbrandization</u>)

$$T \models A$$
$$\Updownarrow$$
$$T^S \models A^H$$

purely universal theory

Π_2-formula.

Generalized Herbrand's thm.
end

<u>Second (original form of the Herbrand's thm)</u>

Assume all the propositional quantifiers of A are $\wedge, \vee, \neg$, and $\neg$ appears only before atomic subformulas.

An <u>$\vee$-expansion</u> of A is generated by subsequent replacements of subformulas B by $B \vee B$.

A <u>strong</u> <u>$\vee$-expansion</u> is the one where one applies the above procedure only to subformulas of the form $\exists y B(y)$.

<u>Prenexification</u> of A is generated by renaming all the bound variables so that each one is quantified only once and then putting A into a prenex normal form.

E.g. $\exists x A(x) \vee \exists x B(x)$

$$\rightsquigarrow$$

$\exists x A(x) \vee \exists y B(y)$

$\vdots$

prenex normal form

Herbrand's proof

Let A be a result of some prenexification.
Assume A is of the form:

$$\forall x_1, \dots, x_{n_1} \exists y_1 \forall x_{n_1+1}, \dots, x_{n_2} \exists y_2 \dots \exists y_r \forall x_{n_r+1}, \dots x_{n_{r+1}} B(\bar{x}, \bar{y}),$$

B - quantifier-free, $0 \leq n_1 \leq n_2 \leq \dots \leq n_{r+1}$.

A <u>witnessing substitution</u> for A is a sequence of semiterms $t_1, \dots, t_r$ s.t.

① each t_i contains arbitrary free variables but only bound variables from $x_1, \dots, x_{n_i}$;

② the formula $B(\bar{x}, t_1, \dots, t_r)$ is a quasitautology.

A <u>witnessing substitution over a theory</u> T is defined as above with ② replaced by

②' $T \models \forall \bar{x}\, B(\bar{x}, \bar{t})$.

An <u>Herbrand proof</u> of A is a prenexification A^* of a strong $\vee$-expansion of A plus a witnessing substitution σ for A^*

Herbrand's proof
cont.

An <u>Herbrand T-proof</u> is defined analogously.

<u>Thm</u>: A f-la A is valid iff
A has an Herbrand proof.

More generally, if T is a universal theory, then:

$$T \models A$$
iff
A has an Herbrand
T-proof.

Proof is done by inspecting LK cut-free proof of A

There is an important 'no-counterexample interpretation' which is used in bounded arithmetic.

Interpolation

Let A, B be so that $A \supset B$ is valid. An <u>interpolant</u> C is a formula such that $A \supset C$, $C \supset B$ are valid and C contains only non-logical symbols which appear in both A and B.

Assume our logic is augmented to include two new constant symbols $\top$ and $\bot$ together with two new initial sequents $\longrightarrow \top$ and $\bot \longrightarrow$.

Write $L(A)$ to denote the set of all non-logical symbols occuring in A plus all the free variables from A. For a cedent Π, $L(\Pi)$ is defined analogously.

<u>Thm: (Craig's Interpolation)</u>

ⓐ Assume $\models A \supset B$. Then, there is a f-la C, s.t. $L(C) \subseteq L(A) \cap L(B)$ and such that $\models A \supset C$ and $\models C \supset B$.

ⓑ Suppose $\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2$ is valid. Then, there is a f-la C, s.t. $L(C) \subseteq L(\Gamma_1, \Delta_1) \cap L(\Gamma_2, \Delta_2)$ and so that $\Gamma_1 \longrightarrow \Delta_1, C$ and $C, \Gamma_2 \longrightarrow \Delta_2$

Interpolation
cont.

Pr: Note that (b) implies (a).

Let P be a cut-free proof P of
$$\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2.$$
We then proceed by induction on the number of strong inferences in P to prove there is C with the desired properties so that P_1 is a proof of $\Gamma_1 \longrightarrow \Delta_1, C$ and P_2 is a proof of $C, \Gamma_2 \longrightarrow \Delta_2$.

In fact, P_1 and P_2 are cut-free and have lengths bounded by a linear factor of the length of P.

For simplicity, assume there are no function symbols and the final strong inference of P is on $\exists$:right with principal f-la in Δ_2. I.e. P ends with:

$$\frac{\begin{array}{c}\vdots\\ \Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2', A(t)\end{array}}{\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2', \exists x A(x)}$$

10.

Interpolation
end.

Since no function symbols appear in the language, t is a constant or a variable.
Induction hypothesis state, that there is an interpolant $C(t)$, s.t.

$\Gamma_1 \rightarrow \Delta_1, C(t)$ and $C(t), \Gamma_2 \rightarrow \Delta_2, A(t)$

are both provable. We define C^* as follows:
if t does not appear in $\Gamma_2 \rightarrow \Delta_2$, then
C^* is $\exists y\, C(y)$.
Otherwise, if t does not appear in $\Gamma_1 \rightarrow \Delta_1$,
then C^* is $\forall y\, C(y)$ and if t appears in both sequents, then C^* is just C.

It can then be checked that

$\Gamma_1 \rightarrow \Delta_1, C^*$ and $C^*, \Gamma_2 \rightarrow \Delta_2, \exists x A(x)$

are LK-provable.

The other cases are done analogously

■

11.

Beth's definability

Let P and P' be predicate symbols with the same arity. Let $\Gamma(P)$ be a set of first-order sentences not involving P'. Denote $\Gamma(P')$ the same set of sentences with every occurence of P replaced with P'.

The set $\Gamma(P)$ is said to <u>explicitly define</u> the predicate P if there is a formula $A(\bar{c})$ s.t. $\Gamma(P) \vdash \forall \bar{x}\, A(\bar{x}) \leftrightarrow P(\bar{x})$.

The set $\Gamma(P)$ is said to <u>implicitly define</u> the predicate P if

$$\Gamma(P) \cup \Gamma(P') \models \forall \bar{x}\, P(\bar{x}) \leftrightarrow P'(\bar{x}),$$

<u>Thm (Beth's Definability)</u>:

$\Gamma(P)$ implicitly defines P iff it explicitly defines P.

<u>Pr</u>: Explicit definability readily implies implicit definability.

For the converse, assume $\Gamma(P)$ is a single sentence (by compactness). Then,

$\Gamma(P) \wedge P(\bar{c}) \models \Gamma(P') \supset P'(\bar{c})$. By Craig interpolation, there is $A(\bar{c})$ s.t.

$\Gamma(P) \wedge P(\bar{c}) \models A(\bar{c})$ and $A(\bar{c}) \models \Gamma(P') \supset P'(\bar{c})$. ∎