Cut Elimination

for first-order sequent calculus LK

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GOAL: <u>CONSTRUCTIVE</u> PROOF OF THE CUT ELIMINATION THEOREM

I.E. EFFECTIVE PROCEDURE FOR <u>CONVERTING</u> A GENERAL LK-PROOF INTO A CUT-FREE LK-PROOF

+ UPPER BOUND ON THE SIZE OF CUT-FREE LK-PROOF <u>IN TERMS OF</u> THE SIZE OF A GIVEN GENERAL LK-PROOF

BY MODIFYING THE CONSTRUCTION

$\rightsquigarrow$ FREE-CUT FREE PROOFS IN LK_e OR LK_σ

DEFINITION: THE *DEPTH*, $dp(A)$, OF A FORMULA IS DEFINED TO EQUAL THE HEIGHT OF A TREE REPRESENTATION OF THE FORMULA

$$dp(A) = 0 \qquad \text{FOR } A \text{ ATOMIC}$$

$$dp(A \wedge B) = dp(A \vee B) = dp(A \supset B) = 1 + \max\{dp(A), dp(B)\}$$

$$dp(\neg A) = dp((\exists x)A) = dp((\forall x)A) = 1 + dp(A)$$

THE *DEPTH* OF A CUT INFERENCE IS DEFINED TO EQUAL THE DEPTH OF ITS CUT FORMULA

CUT FORMULA

$$\frac{\Gamma \longrightarrow \Delta, A \qquad A, \Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta}$$

THEOREM: CUT-ELIMINATION THEOREM

LET P BE A LK-PROOF AND SUPPOSE EVERY CUT FORMULA IN P HAS DEPTH $\leq d$.

THEN THERE IS A CUT-FREE LK-PROOF P^* WITH THE SAME ENDSEQUENT AS P, WITH SIZE

$$\|P^*\| < 2_{2d+1}^{\|P\|}.$$

DEFINITION: THE SUPEREXPONENTIATION FUNCTION 2_i^x, $i, x \geq 0$ IS DEFINED BY

$$2_0^x = x$$

$$2_{i+1}^x = 2^{2_i^x}$$

(I.E. $2_i^x = \left.2^{2^{\cdot^{\cdot^{\cdot^{2^x}}}}}\right\} i$)

NOTES TO CUT-ELIMINATION THEOREM

- ORIGINAL PROOF BY GENTZEN 1935

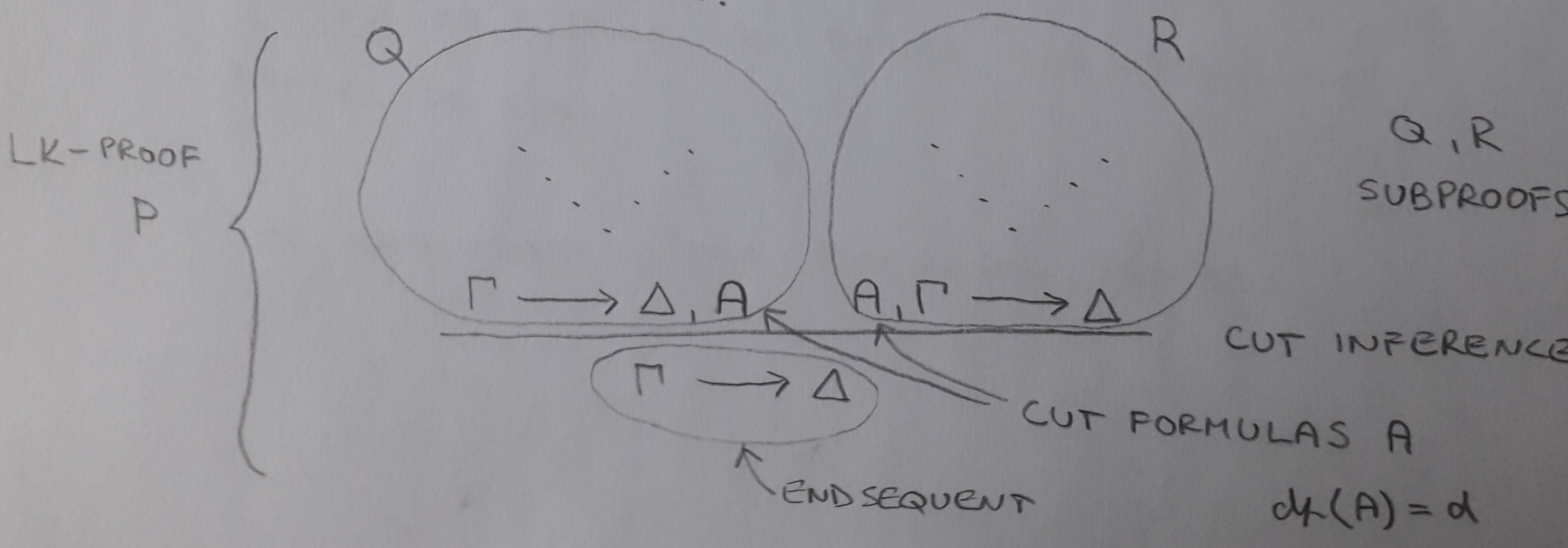
  MAKING LOCAL CHANGES TO A PROOF TO REDUCE THE DEPTH OF CUTS, THE NUMBER OF CUTS, OR THE "RANK OF A CUT".

- BUSS PRESENTS THE PROOF BASED ON MAKING GLOBAL CHANGES TO A PROOF TO REDUCE THE DEPTH OR NUMBER OF CUTS

THE MAIN STEP IS THE FOLLOWING LEMMA

LEMMA: LET P BE AN LK-PROOF WITH FINAL INFERENCE A CUT OF DEPTH d SUCH THAT EVERY OTHER CUT IN P HAS DEPTH STRICTLY LESS THEN d.

THEN THERE IS AN LK-PROOF P^* WITH THE SAME ENDSEQUENT AP P WITH ALL CUTS IN P^* OF DEPTH $< d$ AND WITH $\|P^*\| < \|P\|^2$.



PROOF OF THE LEMMA: (ASSUME P BE IN FREE VARIABLE NORMAL FORM)

- BY CASES ON THE OUTERMOST LOGICAL CONNECTIVE OF A
(BASE CASE = A ATOMIC FORMULA, CASES $\neg, \wedge, \vee, \supset, (\forall x), (\exists x)$)

- WORK WITH THE NOTION OF <u>DIRECT ANCESTOR</u>

(DIRECT) ANCESTOR IS A RELATION ON <u>OCCURENCES</u> OF FORMULAS IN (CEDENTS OF SEQUENTS OF) A PROOF

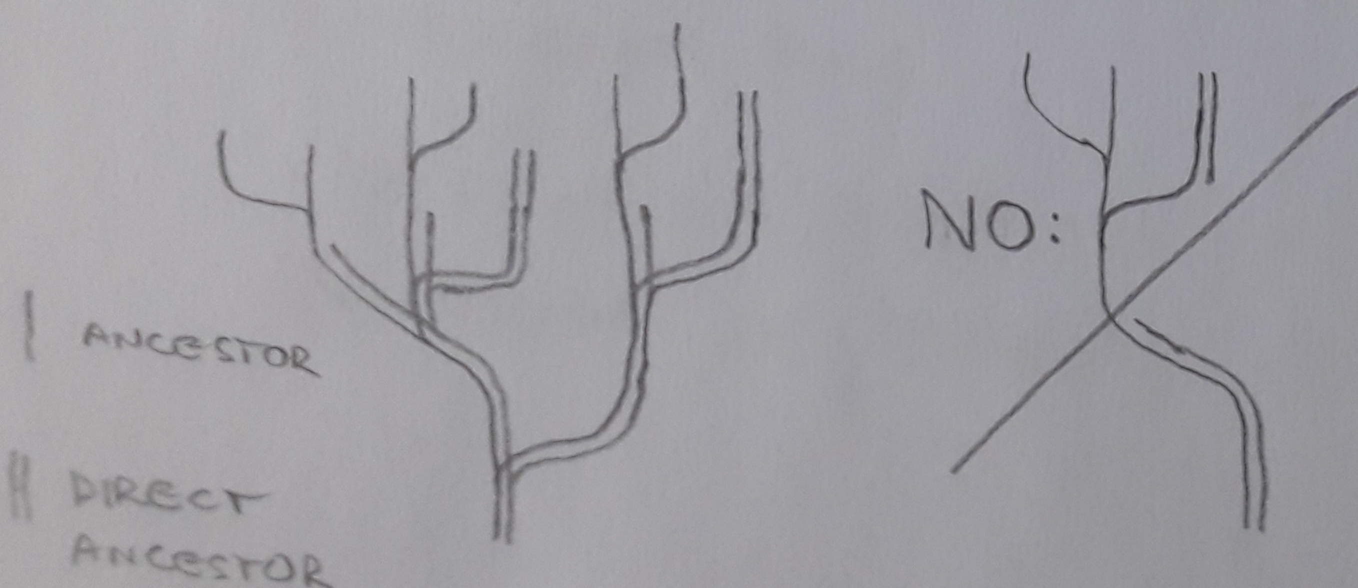
- IMMEDIATE DESCENDENT / ANCESTOR DEFINED FOR EACH INFERENCE RULE

- DESCENDEND / ANCESTOR IS REFLEXIVE TRANSITIVE CLOSURE OR IMMEDIATE DESCENDEND / ANCESTOR

- DIRECT DESCENDEND / ANCESTOR IS A DESCENDENT / ANCESTOR S.T. THEY ARE THE SAME FORMULA

— RECALL THAT IF C IS AN ANCESTOR OF D, THEN C IS A SUBFORMULA OF D

↳ IT FOLLOWS, THAT DIRECT DESCENDENT / ANCESTOR IS A REFLEXIVE TRANSITIVE CLOSURE OF DIRECT IMMEDIATE DESCENDENT / ANCESTOR



— PROPERTIES OF THE RELATION OF DIRECT ANCESTOR :

- DO NOT CROSS THE SEQUENT ARROW
- PRESERVED BY SIDE FORMULAS
- PRESERVED BY PRINCIPAL FORMULAS OF

  EXCHANGE : LEFT / RIGHT

  CONTRACTION : LEFT / RIGHT (BRANCHING) INFERENCES
- DISCONTINUED BY PRINCIPAL FORMULAS OF

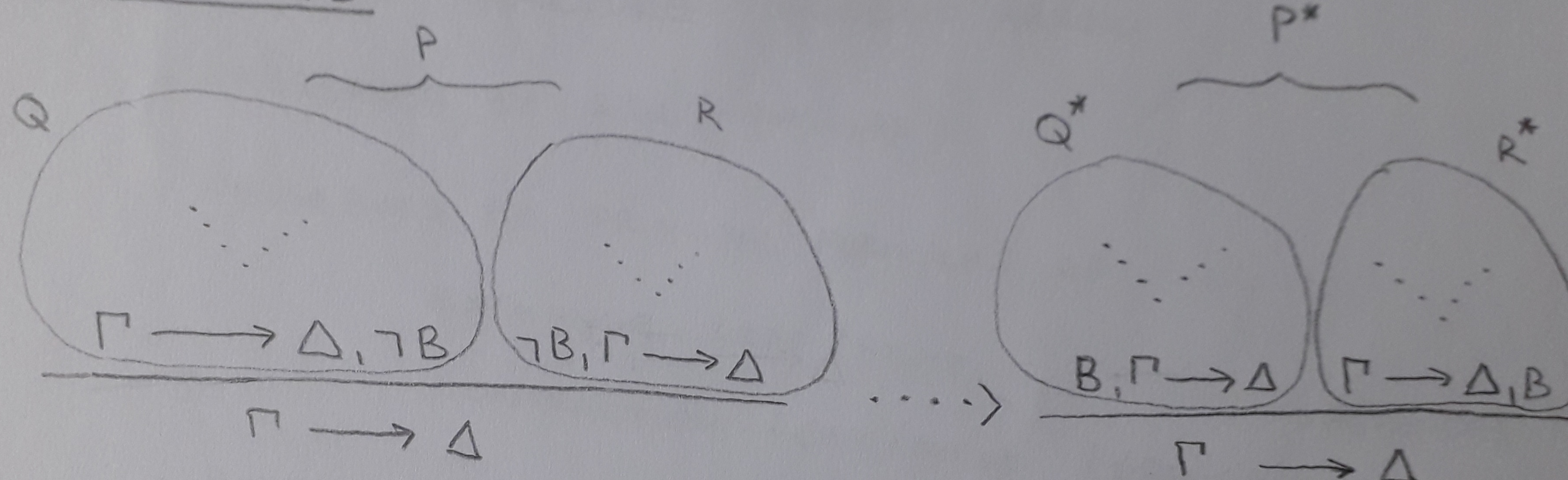
  WEAKENING : LEFT / RIGHT

  ALL PROPOSITIONAL INFERENCES

  BOTH QUANTIFIERS

PROOF OF THE LEMMA

CASE $A = \neg B$:



$$dp(B) = dp(\neg B) - 1$$

MODIFICATION $Q \cdots> Q' \cdots> Q^*$ THE SAME WITH $R \cdots> R^*$

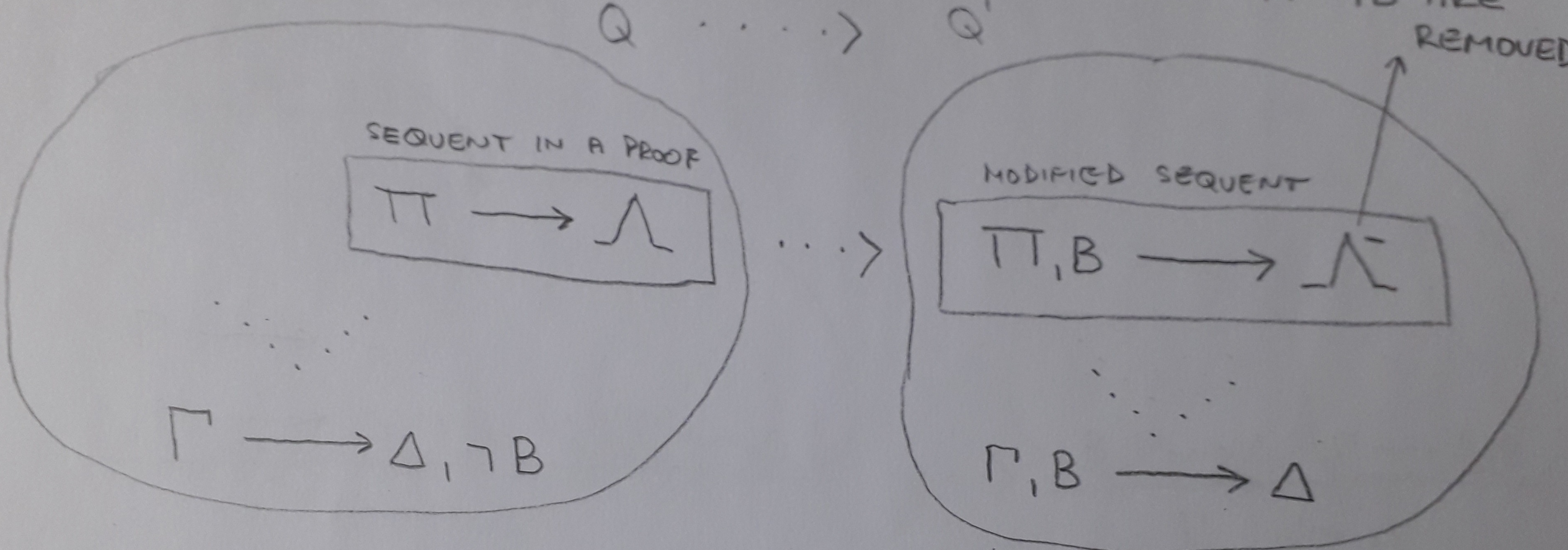
GLOBAL CHANGES

NOT A VALID PROOF POSSIBLY

LOCAL REPAIRS

(CASE $A = \neg B$)

DIRECT ANCESTORS OF $A = \neg B$ ARE REMOVED



NOT NECESSAIRLY A PROOF

WHAT HAPPENS WITH INFERENCES ?

$$\frac{\Pi \longrightarrow \Lambda}{\Sigma \longrightarrow \Phi} \quad \cdots\cdots> \quad \frac{\Pi, B \longrightarrow \Lambda^-}{\Sigma, B \longrightarrow \Phi^-}$$

VALID INFERENCE IN P

NOT NECESSAIRLY A VALID INFERENCE

(CASE $A = \neg B$)

(THERE CAN BE TWO UPPER SEQUENTS)

$$\frac{\Pi \longrightarrow \Lambda}{\Sigma \longrightarrow \Phi} \quad \cdots\rangle \quad \frac{\Pi, B \longrightarrow \Lambda^-}{\Sigma, B \longrightarrow \Phi^-} \quad \cdots\rangle \quad \frac{\Pi, B \longrightarrow \Lambda^-}{\Sigma, B \longrightarrow \Phi^-}$$

- B's IN ANTECEDENT CAN BE CONSIDERED AS PART OF THE SIDE FORMULAS
- DIRECT ANTECEDENTS OF $\neg B$ IN SIDE FORMULAS CAUSES NO PROBLEMS (MODIFY SIDE FORMULAS IN LOWER AND UPPER SUCCEDENT IDENTICALLY

$\longrightarrow$ REMAINS TO CONCERN D.A. OF $\neg B$ <u>IN PRINCIPAL FORMULAS</u>