Lecture 12

overview

topics

- overview of the course
- exam problems
- ex's of further model th. areas

two most important facts

Should you eventually remember just one fact from model theory it should be
the most important fact of model th:

COMPACTNESS OF FO:

$$T \text{ satisfiable} \quad \Leftrightarrow \quad T \text{ finitely satisfiable} \quad .$$

If you will, in addition, also remember something from mathematical logic course(s) it ought to be
the most important fact of logic:

COMPLETENESS OF FO:

$$T \text{ satisfiable} \quad \Leftrightarrow \quad T \text{ consistent} \quad .$$

basic rel's

basic relations among structures:

isomorphism $\mathbf{A} \cong \mathbf{B}$

elementary equivalence $\mathbf{A} \equiv \mathbf{B}$.

Fact:
always $\cong \; \Rightarrow \; \equiv$ but not the other way around.

constructions

we studied a structure

- as a member of a class of structures:
  constructions of new structures, relations among struct's
- as a single structure:
  definable sets and functions

elem.classes

Definition - elementary class

A class of L-structures is an elementary class iff it is the class of all models of some L-theory.

We often considered the following elem.classes:

- DLO: dense linear orderings without end-points
- ACF: algebraically closed fields, and ACF_p: ACF of characteristic p
- $Vect_Q$: vector spaces over $\mathbf{Q}$

rel's

further rel'among structures generalizing $\cong$ and $\equiv$:

- substructure and embeddings
- elementary embedding

con's cont'd

back and forth:

back-and-forth as in the proof of Cantor's thm

Ehrenfeucht-Fraisse games

inside

inside a structure:

- definable sets and functions
- QE
  - explicitly removing one $\exists$ at a time
  - model th. sufficient condition
- types: realized and omitted

application's

applications/relations to concepts from other fields:

- (semi)algebraic geometry notions and QE in ACF and RCF
  - constructible sets = definable sets,
    Tarski's QE = Chevalley's thm,
  - semialgebraic sets = definable sets,
    Tarski's QE = Seidenberg's thm,
- Ax-Grothendieck
- 0-1 law for random graphs

topics left out

The topics we covered are the beginning of model theory, by and large 50+ years old. One topic that lead to many future advances is the stability theory, see Marker's book.

Model theory is being developed in various directions, the most significant perhaps being the geometric model theory that allows to develop some geometric reasoning in situations that are more abstract than those usually covered in (semi)algebraic geometry.

Further interesting topics, relating to combinatorics and computational complexity theory, are:

- finite model theory: model theory of finite structures,
- model th. of arithmetic: a study of models of various fragments of Peano arithmetic.

Q1

(1) The completeness theorem (its precise statement), the compactness theorem and a proof of the latter from the former. Applications of the compactness theorem: constructions of non-standard models of the ring of integers and of the ordered real closed field, a proof of the Ax-Grothedieck theorem on injective polynomial maps on the field of complex numbers.

Q2

(2) Skolemization of a theory and the Lowenheim-Skolem theorem. Vaught's test and its applications to theories DLO, ACF_p and to the theory of vector spaces over the field of rationals. From completeness to decidability for recursive theories.

Q3

(3) Countable categoricity of DLO, the Ehrenfeucht-Fraisse games and elementary equivalence of structures. The theory of random graphs and the 0-1 law for first-order logic on finite graphs.

Q4

(4) Quantifier elimination and its proofs for DLO and ACF. The strong minimality of ACF and the o-minimality of RCF (assuming QE for RCE).

Q5

(5) Types, saturated structures and their properties and existence (an example construction via ultraproduct). Omitting types theorem and MacDowell-Specker theorem (without proofs).