Lecture 3

back-and-forth, Ehrenfeucht-Fraisse

topics

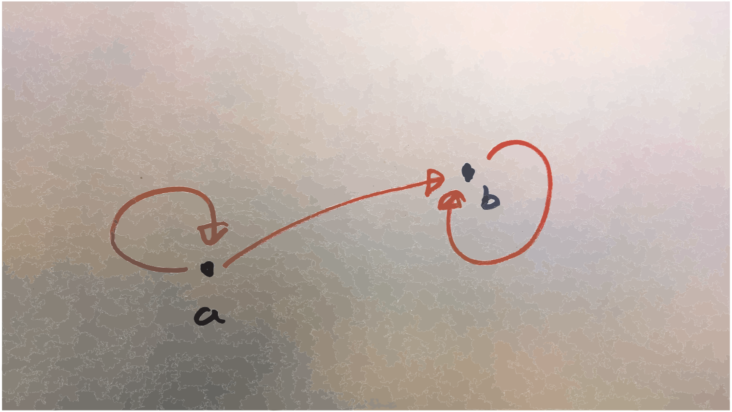
- diagram of a structure
- Cantor's thm
- Ehrenfeucht-Fraisse games
- DLO
- the theory of the countable random graph

leftovers

HW: Show that $Th(\mathbf{A})$ determines $\mathbf{A}$ up to iso if $\mathbf{A}$ is finite.

Our debt: $(Q, <) \preceq (R, <)$.

We shall solve the first task now and the second later in the lecture.



the diagram:

$$R(a,a), R(a,b), \neg R(b,a), R(b,b)\ .$$

Getting rid of the names:

$$\theta \;:=\; \exists x, y\ [x \neq y \wedge R(x,x) \wedge R(x,y) \wedge \neg R(y,x) \wedge R(y,y)]\ .$$

diagram

L, $\mathbf{A}$: any, L_A: L with names for all elements of A

Definition - diagram

The diagram of $\mathbf{A}$, denoted $Diag(\mathbf{A})$, is the set of all

- atomic L_A-sentences true in $\mathbf{A}$, and
- the negations of atomic L_A-sentences false in $\mathbf{A}$.

The diagram is sometimes also called atomic diagram to distinguish it from elementary diagram which is $Th_A(\mathbf{A})$ we introduced last time.

Lemma

$\mathbf{A}$ can be embedded in $\mathbf{B}$ (both L-structures) iff there is an expansion $\mathbf{B}'$ of $\mathbf{B}$ by an interpretation of constants from $L_A \setminus L$ such that

$$\mathbf{B}' \models Diag(\mathbf{A}) \; .$$

ex. of an embedding

L: constant c and binary function symbol $\circ$

A: $(R, 0, +)$

B: $(R_{>0}, 1, \cdot)$

An embedding:

$$a \in A \longrightarrow e^a \in B$$

This works because

$$e^0 = 1 \text{ and } e^{x+y} = e^x \cdot e^y .$$

(Base e could be any, e.g. 2 or 10.)

Cantor's thm

Let us append the earlier definition of theory DLO:

theory DLO (dense linear ordering without endpoints):

- ax's of LO,
- $\forall z \exists x, y (x < z \wedge z < y)$ (no endpoints),
- $\forall x, y (x < y \rightarrow \exists z (x < z \wedge z < y))$ (density).

Cantor's theorem

DLO is countably categorical: all countable models of DLO are isomorphic to $(Q, <)$.

Prf.:
Let **A** and **B** be two countable models of DLO. We can enumerate their universes:

A: $a_0, a_1, \ldots$
B: $b_0, b_1, \ldots$

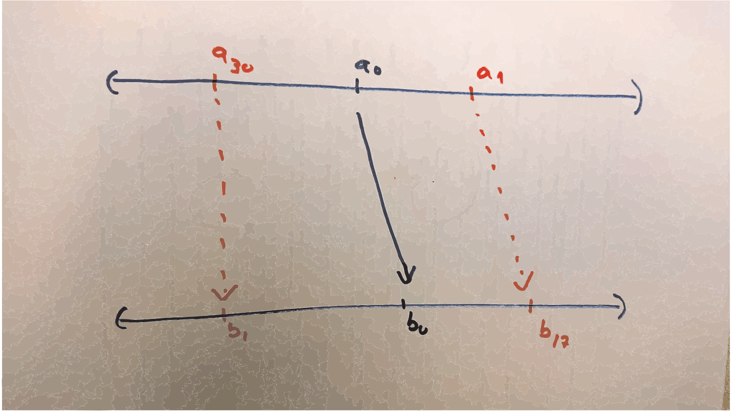
and construct an increasing sequence of partial isomorphisms:

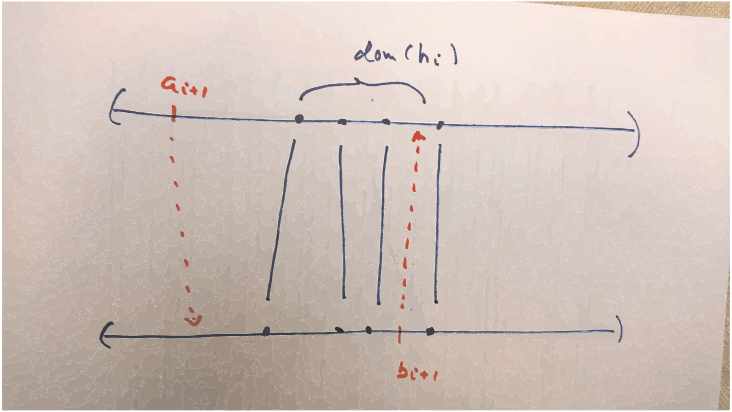
$$h_i :\subseteq A \to B$$

such that

$$\{a_0, \ldots, a_i\} \subseteq dom(h_i) \text{ and } \{b_0, \ldots, b_i\} \subseteq rng(h_i) .$$

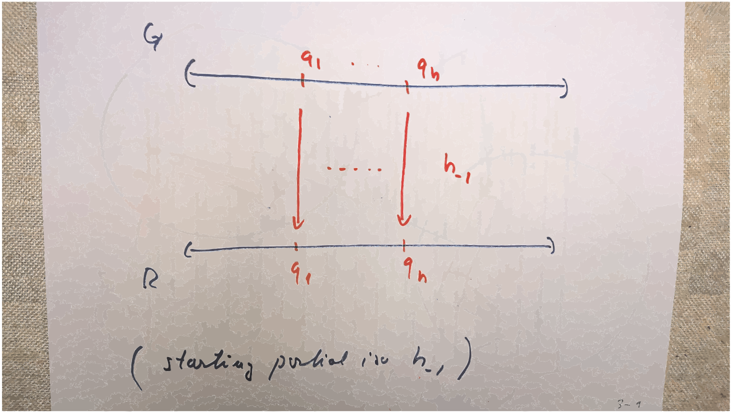
Then $\bigcup_i h_i$ is an isomorphism $\mathbf{A} \cong \mathbf{B}$.


a30
a0
a1
b1
b0
b17

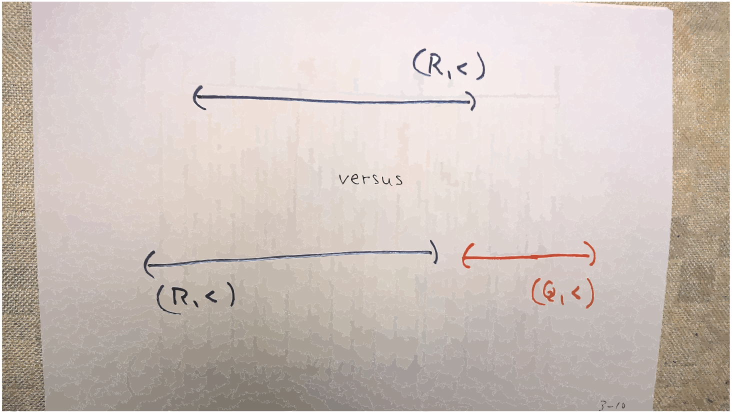


End of the proof.

our debt



uncountable DLOs



Ehrenfeucht-Fraisse games

L: finite, no function symbols
$\mathbf{A}$, $\mathbf{B}$: L-structures (not necessarily finite)

games $G_n(\mathbf{A}, \mathbf{B})$ and $G_\omega(\mathbf{A}, \mathbf{B})$

2 players:

- Spoiler (or $\exists$-player, Eloise),
- Duplicator (or $\forall$-player, Abelard).

moves

1st move:

- Spoiler picks either (i) some $a_1 \in A$ or (ii) some $b_1 \in B$,
- Duplicator replies (i) with some $b_1 \in B$ or (ii) with some $a_1 \in A$.

That is, they determined a pair $(a_1, b_1) \in A \times B$.

$(k+1)$st move:
After first k moves they have already determined pairs $(a_1, b_1), \ldots, (a_k, b_k) \in A \times B$ and proceed as above:

- Spoiler picks either (i) some $a_{k+1} \in A$ or (ii) some $b_{k+1} \in B$,
- Duplicator replies (i) with some $b_{k+1} \in B$ or (ii) with some $a_{k+1} \in A$.

rules

For G_n the game goes on for n rounds, determining

$$(a_1, b_1), \ldots, (a_n, b_n) \in A \times B$$

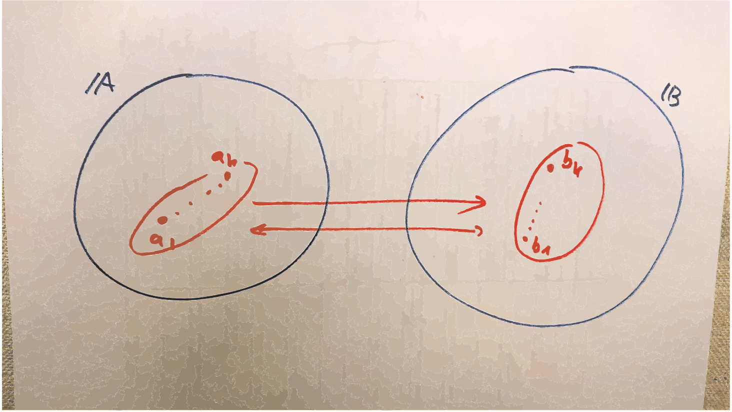
and for G_ω there are infinitely (countably) many rounds, determining

$$(a_1, b_1), \ldots, (a_i, b_i), \ldots \ , \quad \text{all} \quad i \in \mathbf{N} \ .$$

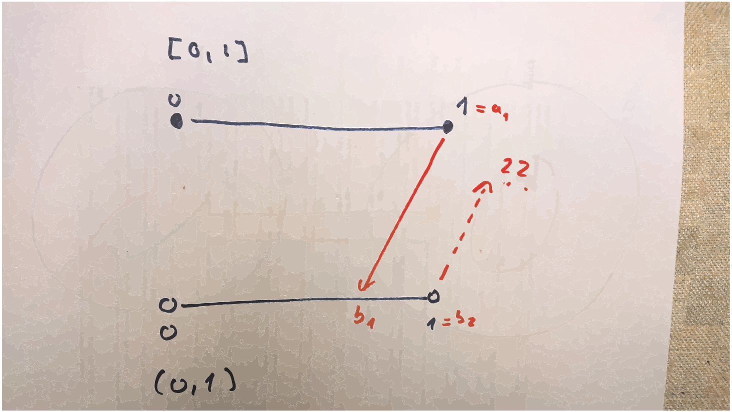
Rules:

- Duplicator wins if the resulting set of pairs is a partial isomorphism from **A** into **B**,
- otherwise Spoiler wins.

D wins



S wins



infinite game

a strategy of a player: any function determining the next move of the player from the history of the play

Lemma

Let **A**, **B** be countable. Then Duplicator has a winning strategy for $G_\omega(\mathbf{A}, \mathbf{B})$ iff $\mathbf{A} \cong \mathbf{B}$.

Prf.: $\Leftarrow$

If $h : A \to B$ is an isomorphism define Duplicator's strategy by:

$$b_k := h(a_k) \quad \text{or} \quad a_k := h^{(-1)}(b_k) \ .$$

Spoiler's strategy

$\Rightarrow$

Spoiler enumerates both universes:

A: $u_0, u_1, \ldots$
B: $v_0, v_1, \ldots$

and in round k plays:

- $k = 2i + 1$ odd: chooses u_i,
- $k = 2i + 2$ odd: chooses v_i.

If Duplicator were to win his answers would form a total isomorphism between **A** and **B**.

□

finite game

Lemma

Let $\mathbf{A}$, $\mathbf{B}$ be countable. Then Duplicator has a winning strategy for $G_k(\mathbf{A}, \mathbf{B})$ for all $k \geq 1$ iff $\mathbf{A} \equiv \mathbf{B}$.

Prf.:
We shall prove only $\Rightarrow$: this is enough for our applications.

Assume $\mathbf{A} \not\equiv \mathbf{B}$ and, in particular,

$$\mathbf{A} \models \theta \quad \text{while} \quad \mathbf{B} \models \neg\theta$$

where θ has the form

$$\forall x_1 \exists x_2 \forall x_3 \ldots Q_k x_k \; \alpha(\overline{x}) \; .$$

Spoiler's strategy

Spoiler: always pick witnesses for $\exists$ quantifier

1st move: $b_1 \in B$ such that

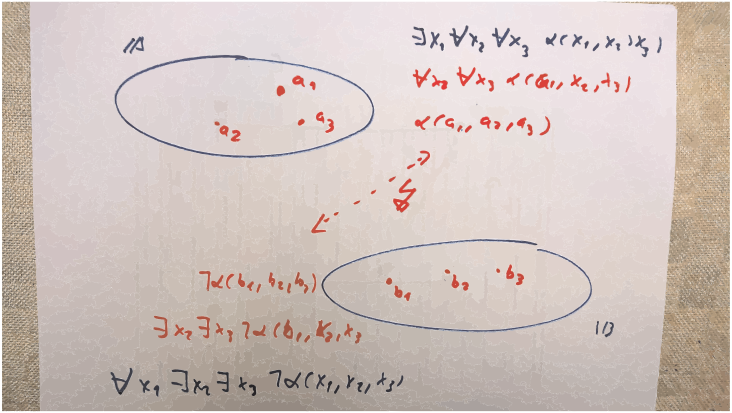
$$\mathbf{B} \models \forall x_2 \exists x_3 \ldots \overline{Q}_k x_k \; \neg\alpha(b_1, x_2, \ldots, x_k)$$

where $\overline{Q}$ is the quantifier opposite to Q.

Key fact: no matter which $a_1 \in A$ Duplicator chooses it will hold:

$$\mathbf{A} \models \exists x_2 \forall x_3 \ldots Q_k x_k \; \alpha(a_1, x_2, \ldots, x_k) \; .$$

Ex.


A
∃x1 ∀x2 ∀x3 α(x1, x2, x3)
∀x2 ∀x3 α(a1, x2, x3)
α(a1, a2, a3)
a1
a2
a3
¬α(b1, b2, b3)
∃x2 ∃x3 ¬α(b1, x2, x3)
∀x1 ∃x2 ∃x3 ¬α(x1, x2, x3)
b1
b2
b3
B

concluding the proof

If Duplicator were to win, the k pairs:

$$(a_1, b_k), \ldots, (a_k, b_k)$$

would form a partial isomorphism while it would also hold:

$$\mathbf{A} \models \alpha(\overline{a}) \text{ and } \mathbf{B} \models \neg\alpha(\overline{b}) \ .$$

That is a contradiction.

□

Rado graph

theory RG: the theory of the countable random graph

language L: binary relation $R(x,y)$

Axioms:

- $\exists x,y\ x \neq y$,
- $\forall x\ \neg R(x,x)$,
- $\forall x,y\ R(x,y) \to R(y,x)$,
- extension axioms, one for each $n \geq 1$:

$$\forall x_1,\ldots,x_n,y_1,\ldots,y_n \bigwedge_{i,j} x_i \neq y_j \ \to\ [\exists z \bigwedge_i R(x_i,z) \wedge \bigwedge_j \neg R(y_j,z)]\ .$$

The 2nd and the 3rd axioms just define undirected graphs without loops.

ext. axioms

categoricity

Lemma

RG has a countable infinite model.

This is a HW problem: construct such a model.

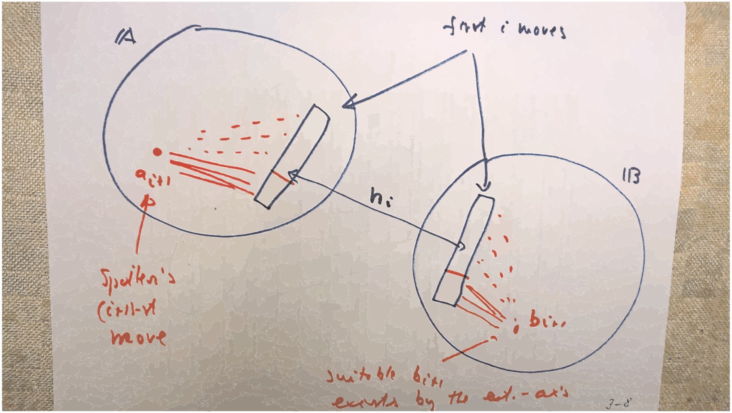
Theorem

RG is countably categorical.

Prf.:
Use G_ω game, as in the next picture.

□

prf



A
first i moves
B
h_i
Spoiler's
(i+1)-st
move
suitable b_{i+1}
exists by the ext.-ax's

determinacy

Are the games determined? I.e. does one of the players alwyas have a winning strategy? Yes for finite games:
Spoiler has a wining strategy iff
- ∃ Spoiler's first move s_1 such that
- ∀ Duplicator's reply d_1 it holds that
- ∃ Spoiler's second move s_2 such that ...
- ...
- Spoiler winns.

and Duplicator has one iff
- ∀ Spoiler's first moves s_1 it holds
- ∃ Duplicator's reply d_1 such that
- ∀ Spoiler''s second move s_2 ...
- ...
- Duplicator winns.

Negations of each other!

ax. of determinacy

For infinite games this argument does not work: we do not have formulas with infinitely many quantifiers.

We could define such flas but would the DeMorgan rules still apply? Are

$$\exists x_1 \forall x_2 \ldots Q_i x_i \ldots\ \alpha(x_1, \ldots)$$

$$\Downarrow \quad \text{negation}$$

$$\forall x_1 \exists x_2 \ldots \overline{Q}_i x_i \ldots\ \neg\alpha(x_1, \ldots)$$

complemenary?

ZFC rules this out as a general rule but it holds for some special games.