To solve Problem 5 our task is to derive all Δ_0-instances of WPHP^{2x}_x from Δ_0-instances of $\mathrm{WPHP}^{x^2}_x$ (the other implication being obvious). That is, if $f : 2x \to x$ violates WPHP^{2x}_x we want to $\Delta_0(f)$-define a map

$$g \ : \ x^2 \to x$$

violating $\mathrm{WPHP}^{x^2}_x$.

Think of f as two injective maps

$$f_0, f_1 \ : \ x \to x$$

with disjoint ranges: $rng(f_0) \cap rng(f_1) = \emptyset$. Simply put:

$$f_0(u) := f(u) \ \text{ if } \ u < x$$

and

$$f_1(u) := f(u + x) \ \text{ if } \ u \leq u < 2x \ .$$

W.l.o.g. we may assume that $x = 2^k$ (because there is always a power of 2 between $2x$ and $4x$ and we could compose h with itself to define a surjection from x onto $4x$), and identify $x^2 = x \times x$ with $x \times \{0,1\}^k$.

A way how to think about the next definition is to picture a depth k binary tree with 2^k different leaves, each hosting a copy of x (i.e. all leaves together represent $x \times x = x^2$). With this idea define map $g : x^2 \to x$ by taking $y < x^2$, identifying it with an ordered pair $(u, i) \in x \times \{0,1\}^k$ where $u < x$ and $i = (i_1, \ldots, i_k) \in \{0,1\}^k$, thinking of it as u being in the copy of x sitting at the leaf which you reach from the root by the path $i_k, \ldots, i_1$, and stipulating that:

$$g(y) \ := \ f_{i_1}(f_{i_2}(\ldots(f_{i_k}(u)\ldots) \ . \qquad (1)$$

You need to draw the binary tree to understand this clearly but basically if you travel from the leaf where (u, i) belongs to towards the root, you start with u and then in succession apply f_0 if you go left and f_1 if you go right.

We need to check two things:

- g is injective,
- the condition in (1) (and hence map g) can be actually defined by a bounded formula.

The first condition is proved by induction on k, assuming we know how to arrange the second condition and so we can also talk about the values along the i path in (1). To arrange the 2nd condition and define the graph $g(y) = z$ we shall need axiom Ω_1. To formalize (1) you say $y = (u, i)$ and

$$\exists s, \ s \text{ is a sequence } s = (s_0, \ldots, s_k) \text{ of length } k + 1$$

s.t.:

$$s_0 = u \wedge \forall t < k, \ s_{t+1} = f_{i_{t+1}}(s_t) \wedge s_k = z \ .$$

Number s codes a sequence of $k + 1 \sim |x|$ numbers $< x$ and hence its bit length is about $|x|^2$ and axiom Ω_1 says exactly that such a big number exists for any x.

You ought to work out the details of this construction.