Now we want to turn the table around and ask if IND proves PHP or any of the variants of WPHP. We considered three PHP-type principles so far:

$$PHP_x^{x+1},\ WPHP_x^{2x}\ \text{ and }\ WPHP_x^{x^2}$$

ordered from logically strongest to weakest (over PA^- implying that $x^2 > 2x > x+1$ for $x \geq 1$). I am writing it with x in place of n in order not to confuse n with a length of any number: in these principles it only plays a role of a parameter.

We shall define actually yet weaker form:

$$WPHP_{|x|-1}^{x}(f) \ \Leftrightarrow\ f \text{ is not a 1-to-1 map from } x \text{ into } |x|-1\ .$$

In other words, the gap between the sizes of the domain and the range is now exponential.

Our next problem is:

Problem 4: Show that $I\Delta_0$ **proves** $WPHP_{|x|}^{x}(f)$ for all Δ_0-definable maps.

For simplicity consider x of the form $x = 2^k$. Hence elements $y < x = 2^k$ (i.e. the domain of f) are - in the sense of the 2nd notes about Problems 2 and 3 - subsets of $\{k-1, \ldots, 0\}$ while the set of $z < |x| - 1 = k$ (containing the range of f) is $\{0, \ldots, k-1\}$; to get exactly this set is the reason why I wrote the cosmetic $|x| - 1$ in the principle as $|x| = \log_2 x + 1$.