Problem 6 (stated below) is the last one in the *positive* (and easier) part of our PHP investigation. The qualification positive means that Problems 1-6 asked us to demonstrate that some statement is provable. The *negative* (and harder in my view) problems then ask to show that something is not provable. Hopefully we get to them sometimes in future.

After Problem 5 we know that versions $\mathrm{WPHP}_x^{2x}$ and $\mathrm{WPHP}_x^{x^2}$ are equivalent over $I\Delta_0 + \Omega_1$ and, even better, that the *exponential* version $\mathrm{WPHP}_{|x|-1}^{x}$ is outright provable (in $I\Delta_0$ alone). Our task, formalized in Problem 6, will be to show that both $\mathrm{WPHP}_x^{2x}$ and $\mathrm{WPHP}_x^{x^2}$ are actually also provable in $I\Delta_0 + \Omega_1$. Our strategy will be to reduce them to something like the exponential version. It is actually possible to reduce them in $I\Delta_0 + \Omega_1$ to $\mathrm{WPHP}_{|x|-1}^{x}$ but that is technically rather cumbersome and, for me at least, not much intuitive. I think it is better to follow the line of thinking how we solved Problem 5.

Namely, assume $g_1 : x^2 \to x$ violates $\mathrm{WPHP}_x^{x^2}$, and assume as before w.l.o.g. that $x = 2^k$, so $x(= \{0, \ldots, x-1\})$ can be identified with binary words of length $k$. And as before, we also think of $x^2$ as of $x \times x$, and analogously of $x^4$ as $x \times x \times x \times x$, etc.

Having $(a, b, c, d) \in x^4$, define $g_2(a, b, c, d) = g_1(g_1(a, b), g_1(c, d))$. Map $g_2$ is an injective map from $x^4$ into $x$. Now the idea is, as in the solution to Problem 5, to iterate this idea along the binary tree of depth $k$, creating injective maps

$$g_t : x^{2^t} \to x \ , \quad \text{for} \quad t = 1, 2, \ldots, k \ .$$

The last one $g_k$ is a map from $x^x$ into $x$ and that would be brought into contradiction as in the original exponential version by Cantor's diagonal argument.

BUT there is a fundamental problem: Parikh's theorem implies that we cannot prove that the number $x^x$ exists at all! We shall try to salvage as much as possible from the earlier argument and the next problem ought to direct you how to do it.

**Problem 6:** *Assume $g : x^2 \to x$ violates WPHP. Assume $x = 2^k$. Show that there is a function $x \times \{0,1\}^k \to x$ sending $(u, i) \in x \times \{0,1\}^k$ to $(u)_i \in x$ such that for any* definable set *$H \subseteq x$ there is $u < x$ s.t. for all $i \in \{0,1\}^k$:*

$$i \in H \quad \textit{iff} \quad (u)_i = 1 \ .$$

*Use this to prove in $I\Delta_0 + \Omega_1$ principles $WPHP_x^{2x}$ and $WPHP_x^{x^2}$.*

Hint: For the first part think of $(u)_i$ as reconstructing the label of the leaf reachable by path $i$ when root is labeled by $u$, and then adopt to this situation Cantor's diagonal argument.