Problem set 18.3.2020

Let $A(y, z)$ be a formula (possibly with other parameters) and let $PHP(A)$ be (the universal closure of):

$$\forall y < x + 1 \exists! z < x A(y, z) \ \rightarrow \ \exists y_1 < y_2 < x + 1 \exists z < x (A(y_1, z) \wedge A(y_2, z)) \ .$$

It expresses the statement that if A defines a graph of a function with domain $\{0, \ldots, x\}$ and range included in $\{0, \ldots, x-1\}$ then it cannot be injective. This is the pigeonhole principle.

Problem 1 Denote by Δ_0-PHP the theory PA^- plus the scheme $PHP(A)$ accepted for all Δ_0-formulas. Show that this theory proves all axioms of $I\Delta_0$, i.e. it proves induction for all bounded formulas.

Now consider (logically weaker) weak forms of the PHP scheme:

- $WPHP^{2n}_n$: replace $x + 1$ in the PHP by $2x$, and
- $WPHP^{n^2}_n$: replace $x + 1$ in the PHP by x^2.

Problem 2: Show the same as in Problem 1 for these two WPHP principles.

Allow in the language of arithmetic also a new binary relation symbol $R(u, v)$: $\Delta_0(R)$ are bounded formulas in this extended language and $I\Delta_0(R)$ is defined as $I\Delta_0$ but in this language. In particular, it has no specific axioms about R: R occurs only in formulas in the induction scheme.

Problem 3: Show that the arguments solving Problems 1 and 2 work for $\Delta_0(R)$ as well.